Automatic Construction of Explicit R Matrices for the One-Parameter Families of Irreducible Typical Highest Weight $(\dot{0}_m|\dot{\alpha}_n)$ Representations of $U_q[gl(m|n)]$

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Abstract

We detail the automatic construction of R matrices corresponding to (the tensor products of) the $(\dot{0}_m|\dot{\alpha}_n)$ families of highest-weight representations of the quantum superalgebras $U_q[gl(m|n)]$. These representations are irreducible, contain a free complex parameter α, and are 2^{mn} dimensional. Our R matrices are actually (sparse) rank 4 tensors, containing a total of 2^{4mn} components, each of which is in general an algebraic expression in the two complex variables q and α.

Although the constructions are straightforward, we describe them in full here, to fill a perceived gap in the literature. As the algorithms are generally impracticable for manual calculation, we have implemented the entire process in MATHEMATICA; illustrating our results with $U_q[gl(3|1)]$.

1 Introduction

Broadly, R matrices are solutions to the various versions of the Yang–Baxter equation, and as such, are of great interest in mathematical physics and knot theory (see, e.g. [18]), both in their algebraic (i.e. "universal") forms, and in their (matrix) representations (i.e. "quantum" forms), useful for explicit computations. Here, we will be specifically concerned with quantum R matrices associated with the quantum superalgebras $U_q[gl(m|n)]$.

Although much is known about the origin and properties of quantum superalgebra R matrices (e.g. [19] provides universal R matrices), explicit examples of their quantum R matrices are rare in the literature, due largely to the computational effort involved in obtaining them. This paper describes the automation of an algorithm to generate a suite of explicit quantum R matrices for $U_q[gl(m|n)]$. As readers of this organ may not be familiar with these algebraic structures, we provide a full description of their details.

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Specifically, we construct *trigonometric* R matrices $\check{R}^{m,n}(u)$ corresponding to the α-parametric highest weight minimal representations labeled $(\dot{0}_m|\dot{\alpha}_n)$ of the $U_q[gl(m|n)]$. These irreducible representations are 2^{mn} dimensional, and contain free complex parameters q and α; the real variable u is a 'spectral' parameter. Quantum R matrices $\check{R}^{m,n}$ are immediately obtainable as the spectral limits $u \to \infty$ of the $\check{R}^{m,n}(u)$.

Our R matrices are in fact *graded*, as they are based on graded vector spaces, hence they actually satisfy *graded* Yang–Baxter equations. However, it is a simple matter to remove this grading and transform them into objects that satisfy the usual Yang–Baxter equations.

The constructions have been implemented in MATHEMATICA, and results obtained for $n = 1$ and $m = 1, 2, 3, 4$; we illustrate the algorithms using $U_q[gl(3|1)]$. Full listings of all our R matrices have been announced in [6].

As they are solutions to Yang–Baxter equations, our R matrices are of immediate practical interest. Firstly, they are of physical interest in that they are applicable to the construction of exactly solvable models of interacting fermions. Corresponding to $\check{R}^{m,1}(u)$, we may construct an integrable 2^m state fermionic model on a lattice. Models associated with $U_q[gl(2|1)]$ and $U_q[gl(3|1)]$ have been discussed in [14] and [13], respectively. The $U_q[gl(4|1)]$ case has an elegant interpretation in terms of a 2-leg ladder model for interacting electrons: a discussion of this is provided in [6].

Furthermore, corresponding to each $\check{R}^{m,n}$, we may obtain a polynomial 'Links–Gould' link invariant $LG^{m,n}$ [21], cf. the celebrated Jones polynomial. These $LG^{m,n}$ are two-variable, integer-coefficient Laurent polynomials, and are generally substantially more powerful than the Jones polynomial in distinguishing knots. ($LG^{1,1}$ degenerates to the well-known Alexander–Conway polynomial in the single variable $q^{2\alpha}$ (cf. [2]).) A fuller documentation of the suite $LG^{m,n}$ has been provided by myself in collaboration with Louis Kauffman and Jon Links in [4, 5, 7, 9]. Although the $LG^{m,n}$ are far from being complete invariants, as they can distinguish neither mutants nor inversion [5, 9], it turns out that even $LG^{2,1}$ is in fact more powerful than the well-known two-variable HOMFLY and Kauffman invariants, being able to distinguish (including chirality) all prime knots of up to 10 crossings [5]. Their evaluation also involves automatic symbolic computation, but the computational aspects are comparatively pedestrian.

Lastly, we mention explicitly that this paper contains no new theorems, although it does contain two new technical lemmas, proven in Appendix A. It is primarily intended to provide a proper foundation for the results presented in [6, 7], although it also serves as a tutorial on an application of symbolic computation. Whilst it specifically pertains to representations of $U_q[gl(m|n)]$, many of the algorithms have a much broader application.

The following subsections provide a synopsis of the paper.

1.1 Algebraic overview

Fixing m and n, we are initially interested in a 2^{mn} dimensional vector space V that is a module for the $U_q[gl(m|n)]$ minimal typical highest weight representation $\Lambda = (\dot{0}_m|\dot{\alpha}_n)$. The algebra contains a free complex variable q, whilst the representation π_Λ acting on V contains a free complex variable α. Our V is actually $(\mathbb{Z}_2)$ *graded*; this ensures compatibility with the $(\mathbb{Z}_2)$ grading of $U_q[gl(m|n)]$.

Using the properties of $U_q[gl(m|n)]$, we apply a version of the *Kac induced module construction* (KIMC) [16, 17] to establish a (*weight*) basis $\{|i\rangle\}_{i=1}^{2^{mn}}$ for V. This involves postulating $|1\rangle$ as a highest weight vector, and recursively acting on $|1\rangle$ with all possible distinct products of simple lowering generators $E^{a+1}{}_a$ to define the other basis vectors, normalising as we go. This construction requires a 'Poincaré–Birkhoff–Witt (PBW) lemma' for $U_q[gl(m|n)]$ [8, 28], i.e. a set of commutations sufficient to transform any product of algebra generators into a normal form (see §4.2), together with a statement that the algebra is spanned by the set of all such normal forms.

Where V has a graded weight basis $\{|i\rangle\}_{i=1}^{2^{mn}}$, the tensor product module $V \otimes V$ has a natural 2^{2mn} dimensional basis $\{|i\rangle \otimes |j\rangle\}_{i,j=1}^{2^{mn}}$, which inherits a weight system and a grading from V. For our particular representation, the orthogonal decomposition of $V \otimes V$ is known [11], and contains no multiplicities, viz:

$$V \otimes V = \bigoplus_k V_k,$$

where the submodule V_k has highest weight λ_k, and these λ_k are known, and all distinct. To build R matrices acting on $V \otimes V$, we require an alternative, orthonormal weight basis $\mathfrak{B} = \bigcup_k \mathfrak{B}_k$ for $V \otimes V$, corresponding to this decomposition, viz $\mathfrak{B}_k$ is a basis for V_k. Again using the KIMC, the basis vectors of each $\mathfrak{B}_k$ are derived as linear combinations of the form $\theta_{ij}(|i\rangle \otimes |j\rangle)$, where the coefficients θ_{ij} are algebraic expressions in q and α. This process initially yields a basis $\overline{\mathfrak{B}}_k$ that is not necessarily orthonormal, so we also apply a Gram–Schmidt process to orthonormalise $\overline{\mathfrak{B}}_k$ into $\mathfrak{B}_k$. The desired R matrix is then a weighted sum of projectors onto these V_k, where the weights are eigenvalues of the appropriate second order Casimir invariants.

The algebraic structure of $U_q[gl(m|n)]$ is detailed in §2, and an introduction to its highest weight representations is provided in §3. In §4, we provide a normal ordering and a PBW lemma for $U_q[gl(m|n)]$. The construction of our particular $(\dot{0}_m|\dot{\alpha}_n)$ representations is detailed in §5. §6 describes the construction of the bases $\mathfrak{B}_k$, and §7 describes the construction of projectors and R matrices.

1.2 Implementation and results

Explicit computations within the representation theory of quantum superalgebras are tedious and error-prone when performed manually. The dimensions of representations are generally large, and in our case, we have the presence of the two variables q and α; these generally manifest themselves in complicated rational algebraic expressions, whose symmetries must be continually identified and exploited to avoid the arising of intractable messes of algebra.

The construction of the basis $\{|i\rangle\}_{i=1}^{2^{mn}}$ involves many applications of the PBW lemma to simplify long strings of algebra generators. This is computationally expensive; firstly as the simplification involves a minimally-efficient sorting process, and second as it involves a geometric explosion in the number of terms being sorted.

The construction of the weight space bases $\mathfrak{B}_k$ is nontrivial, as each basis vector of each $\mathfrak{B}_k$ generally contains many terms of the form $\theta_{ij}(|i\rangle \otimes |j\rangle)$, where the coefficients θ_{ij} are generally complicated rational algebraic expressions in q and α. (That said, we have avoided the more theoretically difficult situation of computing weight space decompositions in cases where there are weight multiplicities in the underlying carrier space V.) Although the R matrices have 2^{4mn} components, Nature is kind to us in that most of these components are zero, and those that are not are generally simpler than the θ_{ij}.

To the best of our knowledge, computer implementation of the algebraic structures and algorithms described herein has not previously been achieved. We have implemented the entire process as a suite of MATHEMATICA functions; the thousands of lines of code perform algebraic computations that a human being could not ever realistically expect to perform correctly.

From §2.3 onwards, we use $U_q[gl(3|1)]$ to illustrate our results. These are summarised in Appendix B; where we list the explicit matrix elements for the generators of the underlying 8 dimensional representation, orthonormal bases $\mathfrak{B}_k$ for the 4 submodules $V_k \subset V \otimes V$, the components of the associated 4 projectors P_k onto the V_k, and finally, the trigonometric and quantum R matrices, $\check{R}^{3,1}(u)$ and $\check{R}^{3,1}$, respectively.

Whilst there are no theoretical limits to m and n, a current practical limit for computation is $mn \leqslant 4$. This is convenient, as an immediate application [6] of the material critically requires $\check{R}^{4,1}(u)$. Although translation of the interpreted MATHEMATICA code into a compiled language would increase the speed of the computations, storage requirements would still limit mn to perhaps 7 in the general case.

Further discussion of implementational issues and results is provided in §8.

2   The quantum superalgebras $U_q[gl(m|n)]$

The algebraic structures labeled $U_q[gl(m|n)]$ are *quantum superalgebras*,[1] described in many places, e.g. [1, 10, 24, 25, 26], and in the book [3, see §6.5].

For our purposes, m and n are positive integers, to be regarded as *fixed*, and q is to be regarded as a nonzero complex variable. As $U_q[gl(m|n)]$ may be unfamiliar to the readers of this organ, in §2.1 we introduce its phylogeny, and in §2.2, we provide a full description of its structure in terms of generators and relations. Beyond that, in §2.3 we describe its root system, and in §2.4 we show how it may be regarded as a Hopf (super)algebra.

2.1   The phylogeny of $U_q[gl(m|n)]$

1. Where n is a positive integer, recall that the *Lie algebra $gl(n)$* is equivalent to the usual (complex) vector space of $n \times n$ (complex) matrices augmented by a 'vector multiplication' operation which is the usual matrix multiplication. $gl(n)$ is of course a unital algebra, and is of dimension n^2 and rank $n-1$. The n^2 generators $\{e^a{}_b\}_{a,b=1}^n$ of $gl(n)$ satisfy a *commutation relation*:

    $$[e^a{}_b, e^c{}_d] = \delta^c_b e^a{}_d - \delta^a_d e^c{}_b,$$

    where $[\cdot,\cdot]$ is the usual *commutator (bracket)*, defined for $X, Y \in gl(n)$ by:

    $$[X,Y] \triangleq XY - YX.$$

2. Letting both m and n be positive integers, the *Lie superalgebra $gl(m|n)$* may be obtained from $gl(m+n)$ by retaining the generators $\{e^a{}_b\}_{a,b=1}^{m+n}$, but modifying the definition of the commutator bracket and commutation relations to include some 'parity factors' of ± 1. Specifically, we have the commutation relation:

    $$[e^a{}_b, e^c{}_d] = \delta^c_b e^a{}_d - (-)^{[e^a{}_b][e^c{}_d]} \delta^a_d e^c{}_b, \tag{1}$$

    where $[\cdot,\cdot]$ is now the *graded commutator (bracket)*, defined for homogeneous (see below) $X, Y \in gl(m|n)$ by:

    $$[X,Y] \triangleq XY - (-)^{[X][Y]} YX, \tag{2}$$

    and extended by linearity. In both (1) and (2), $[X] \in \{0,1\}$ refers to the *grading* of the homogeneous element X. For this reason, Lie superalgebras are sometimes called "graded Lie algebras". From the $gl(m+n)$ case, we see that $gl(m|n)$ is of dimension $(m+n)^2$ and rank $m+n-1$.

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[1]These structures are sometimes called "quantum super*groups*", but they *are* actually (associative, noncommutative) algebras.

3. $U[gl(m|n)]$ is then the usual *universal enveloping algebra* obtained from $gl(m|n)$ by regarding the $gl(m|n)$ generators as letters in words contained in $U[gl(m|n)]$, where the (graded) commutator bracket acts as a relation to reduce the algebra somewhat. $U[gl(m|n)]$ is infinite dimensional, although of finite rank, viz, again $m + n - 1$.

4. The *quantum superalgebra* $U_q[gl(m|n)]$ is then a so-called 'q-deformation'[2] of $U[gl(m|n)]$, which maintains its viability as a Hopf (super)algebra structure (see below) [12]. Roughly speaking, the deformation amounts to 'exponentiation by q'; indeed $U[gl(m|n)]$ may be recovered as the limit $q \to 1$ of $U_q[gl(m|n)]$. $U_q[gl(m|n)]$ is of course also infinite dimensional, and again of rank $m + n - 1$.

2.2 Generators and relations for $U_q[gl(m|n)]$

Following Zhang [28, pp1237-1238], we provide a full description of $U_q[gl(m|n)]$ in terms of generators and relations. For various invertible X, we will repeatedly use the notation $\overline{X} \triangleq X^{-1}$.

2.2.1 $U_q[gl(m|n)]$ generators

Where $\mathcal{I} \triangleq \{1, \ldots, m + n\}$ is the set of the $gl(m|n)$ *indices*, we define a $\mathbb{Z}_2$ grading $[\cdot] : \mathcal{I} \to \mathbb{Z}_2$:

$$[a] \triangleq \begin{cases} 0 & \text{if } a \leqslant m & \text{even indices} \\ 1 & \text{else} & \text{odd indices.} \end{cases}$$

Throughout, we shall use dummy indices $a, b \in \mathcal{I}$ where meaningful. A set of $(m + n)^2$ generators for $U_q[gl(m|n)]$ is then:

$$\left\{ \begin{array}{lll} K_a, & m + n & \text{Cartan} \\ E^b{}_a, \quad a < b & \frac{1}{2}(m + n)(m + n - 1) & \text{lowering} \\ E^a{}_b, \quad a < b & \frac{1}{2}(m + n)(m + n - 1) & \text{raising} \end{array} \right\}. \tag{3}$$

Let us now introduce the notation, for any $a \in \mathcal{I}$:

$$q_a \triangleq q^{(-)^{[a]}},$$

For any power N, replacing q with q^N immediately shows that $(q_a)^N = (q^N)_a$, so we may write q_a^N with impunity, specifically, we will write $\overline{q}_a \equiv q_a^{-1}$. Next, an equivalent notation for K_a is $q_a^{E^a{}_a}$; where the exponential is defined in the usual manner as an infinite sum, thus powers K_a^N are meaningful; specifically, we will often be working with $N \in \frac{1}{2}\mathbb{Z}$. Thus, under the mapping $q \mapsto \overline{q}$, K_a is mapped to $\overline{K}_a$, where we intend $\overline{K}_a \equiv K_a^{-1}$. As expected, for arbitrary powers M, N, we have:

$$K_a^M K_a^N = K_a^{M+N} \quad \text{where} \quad K_a^0 \equiv \text{Id},$$

where Id is the $U_q[gl(m|n)]$ identity element. Apart from $N \in \mathbb{N}$, powers (i.e. products) of the non-Cartan generators $(E^a{}_b)^N$ for $a \neq b$, are not meaningful.

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[2]We might say 'quantum deformation' here but the relation to quantum mechanics is more of analogy than of rigor.

On the generators we define a natural $\mathbb{Z}_2$ grading in terms of the grading on the indices:

$$[K_a^N] \triangleq 0, \qquad\qquad [E^a{}_b] \triangleq [a] + [b] \pmod 2, \qquad\qquad (4)$$

where the former may be seen as a special case of the latter by setting $a = b$ and making the identification $K_a \equiv q^{(-)^{[a]}E^a{}_a}$. We use the terms "even" and "odd" for generators in the same manner as we do for indices. Elements of $U_q[gl(m|n)]$ are said to be *homogeneous* if they are linear combinations of generators of the same grading. The product XY of homogeneous $X, Y \in U_q[gl(m|n)]$ has grading:

$$[XY] \triangleq [X] + [Y] \pmod 2. \qquad\qquad (5)$$

Thus, for example, inspection of (4) and (5) shows that we may cheerfully substitute $[E^a{}_c]$ for $[E^a{}_b E^b{}_c]$. Further, we also have the following useful results for $a < b < c < d$:

$$[E^a{}_d][E^b{}_c] = [E^b{}_c] \qquad \text{and} \qquad [E^a{}_b][E^b{}_c] = [E^a{}_b][E^c{}_d] = 0.$$

2.2.2 $U_q[gl(m|n)]$ simple generators

The full set of generators (3) includes some redundancy; some can be regarded as *simple* in that the rest may be expressed in terms of them. We shall call the following subset of $3(m + n) - 2$ generators the $U_q[gl(m|n)]$ *simple* generators:

$$\left\{ \begin{array}{lll} K_a, & m + n & \text{Cartan} \\ E^{a+1}{}_a, & a < m + n & m + n - 1 \quad \text{simple lowering} \\ E^a{}_{a+1}, & a < m + n & m + n - 1 \quad \text{simple raising} \end{array} \right\}.$$

The fact that there are $m + n - 1$ simple lowering generators indicates that $U_q[gl(m|n)]$ has rank $m + n - 1$. Note that there are only two *odd* simple generators: $E^{m+1}{}_m$ (lowering) and $E^m{}_{m+1}$ (raising).

2.2.3 $U_q[gl(m|n)]$ nonsimple generators

In the $gl(m|n)$ case, the remaining nonsimple (non-Cartan) generators satisfy the same commutation relations as the simple generators. The situation is very different for $U_q[gl(m|n)]$; the nonsimple generators do *not* satisfy the same commutation relations as do the simple generators. Instead, they are recursively defined in terms of sums of products of the simple generators (see [27, p1971, (3)] and [28, p1238, (2)]). Strictly speaking, they are not explicitly required for the definition of the algebra; their use can help simplify otherwise large expressions.

To whit, a set of $U_q[gl(m|n)]$ nonsimple generators may be defined recursively for $a < b$ by:

$$\left. \begin{array}{llll} \text{(a)} & E^b{}_a & \triangleq & E^b{}_c E^c{}_a - q_c E^c{}_a E^b{}_c \qquad \text{nonsimple lowering} \\ \text{(b)} & E^a{}_b & \triangleq & E^a{}_c E^c{}_b - \overline{q}_c E^c{}_b E^a{}_c \qquad \text{nonsimple raising,} \end{array} \right\} (6)$$

where $a < c < b$; viz c is an *arbitrary* index, we do *not* intend a sum here.

In §5, we will have use for an alternative set of nonsimple generators, again defined recursively for $a < b$ by:

$$
\left.
\begin{array}{llll}
\text{(a)} & \mathbf{E}^b{}_a & \triangleq & \mathbf{E}^b{}_c\mathbf{E}^c{}_a - \overline{q}_c\mathbf{E}^c{}_a\mathbf{E}^b{}_c \qquad \text{alternative nonsimple lowering} \\
\text{(b)} & \mathbf{E}^a{}_b & \triangleq & \mathbf{E}^a{}_c\mathbf{E}^c{}_b - q_c\mathbf{E}^c{}_b\mathbf{E}^a{}_c \qquad \text{alternative nonsimple raising,}
\end{array}
\right\} (7)
$$

where we intend $\mathbf{E}^a{}_b \triangleq E^a{}_b$ when $E^a{}_b$ is simple, viz, for any $|a - b| = 1$. Note that we use a boldface $\mathbf{E}^a{}_b$ where the original source [27] uses an overline $\overline{E}^a{}_b$; the use of the boldface notation saves the overline for indicating inverses.

These definitions may be written more concisely with some more notation. Writing $S^a_b \triangleq \mathrm{sign}(a - b)$, we may replace (6) and (7), for all $a \neq b$, by:

$$
\left.
\begin{array}{llll}
\text{(a)} & E^a{}_b & = & E^a{}_c E^c{}_b - q_c^{S^a_b} E^c{}_b E^a{}_c \\
\text{(b)} & \mathbf{E}^a{}_b & = & \mathbf{E}^a{}_c\mathbf{E}^c{}_b - \overline{q}_c^{S^a_b}\mathbf{E}^c{}_b\mathbf{E}^a{}_c.
\end{array}
\right\} (8)
$$

The two different sets of generators are in fact Hermitian conjugates. For all meaningful indices a, b, we have:

$$
(E^a{}_b)^\dagger = \mathbf{E}^b{}_a, \qquad (\mathbf{E}^a{}_b)^\dagger = E^b{}_a, \qquad (K^N_a)^\dagger = K^N_a,
$$

and these definitions ensure that $(X^\dagger)^\dagger = X$ for all $U_q[gl(m|n)]$ generators X. Note that these are ordinary, not $\mathbb{Z}_2$ graded Hermitian conjugates, meaning that we have $(XY)^\dagger = Y^\dagger X^\dagger$; expressly *not* $(XY)^\dagger = (-)^{[X][Y]}Y^\dagger X^\dagger$.

Lastly, we mention a result of Zhang [27, Lemma 3], which gives us a more efficient formula than (8b) for expanding the alternative nonsimple generators:

$$
\mathbf{E}^a{}_b = E^a{}_b + S^a_b \sum_c \Delta_c E^c{}_b \mathbf{E}^a{}_c, \tag{9}
$$

for *any* indices $a \neq b$, where the sum is over all c strictly between a and b. (If $|a - b| = 1$, then the sum is ignored, and the result is trivial.)

Note that in (9), we have introduced the following handy notation:

$$
\left.
\begin{array}{ll}
\Delta \triangleq q - \overline{q}, & \Delta_a \triangleq q_a - \overline{q}_a = (-)^{[a]}(q - \overline{q}) = (-)^{[a]}\Delta \\
\overline{\Delta} \triangleq (\Delta)^{-1} & \overline{\Delta}_a \triangleq (\Delta_a)^{-1}.
\end{array}
\right\} (10)
$$

2.2.4 The graded commutator

The *graded commutator* $[\cdot, \cdot] : U_q[gl(m|n)] \times U_q[gl(m|n)] \to U_q[gl(m|n)]$, is defined for homogeneous $X, Y \in U_q[gl(m|n)]$ by (2), viz:

$$
[X, Y] \triangleq XY - (-)^{[X][Y]}YX,
$$

and extended by linearity. For completeness, we mention that for *associative* superalgebras, of which $U_q[gl(m|n)]$ is certainly an example, we have the following useful graded commutator identities:

$$
\left.
\begin{array}{llll}
\text{(a)} & [XY, Z] & = & X[Y, Z] + (-)^{[Y][Z]}[X, Z]Y \\
\text{(b)} & [X, YZ] & = & [X, Y]Z + (-)^{[X][Y]}Y[X, Z].
\end{array}
\right\} (11)
$$

2.2.5 $U_q[gl(m|n)]$ relations

With this notation, we have the following $U_q[gl(m|n)]$ *relations*:

1. The Cartan generators all commute; for any powers M, N:

$$K_a^M K_b^N = K_b^N K_a^M. \tag{12}$$

2. The Cartan generators commute with the simple raising and lowering generators in the following manner:

$$K_a E^b{}_{b\pm1} \overline{K}_a = q_a^{(\delta_b^a - \delta_{b\pm1}^a)} E^b{}_{b\pm1}. \tag{13}$$

From (13), we have the following useful interchange:

$$K_a E^b{}_{b\pm1} = q_a^{(\delta_b^a - \delta_{b\pm1}^a)} E^b{}_{b\pm1} K_a. \tag{14}$$

In Lemma 2 (proved in Appendix A.1), we show that (14) may be much strengthened to:

$$K_a^N E^b{}_c = q_a^{N(\delta_b^a - \delta_c^a)} E^b{}_c K_a^N, \tag{15}$$

for *any* meaningful indices b, c (viz $b < c$, $b > c$, and even $b = c$), and any power N.

3. The non-Cartan simple generators satisfy the following commutation relations (this is the really interesting part!):

$$[E^a{}_{a+1}, E^{b+1}{}_b] = \delta_b^a \frac{K_a \overline{K}_{a+1} - \overline{K}_a K_{a+1}}{q_a - \overline{q}_a}. \tag{16}$$

Alternatively, again employing the notation of (10), we may write this:

$$[E^a{}_{a+1}, E^{b+1}{}_b] = \delta_b^a (-)^{[a]} \left[(-)^{[a]} E^a{}_a - (-)^{[a+1]} E^{a+1}{}_{a+1} \right]_q, \tag{17}$$

where we have introduced the q-bracket, defined for various invertible $X \in U_q[gl(m|n)]$, including scalars (well, scalar multiples of Id):

$$[X]_q \triangleq \frac{q^X - \overline{q}^X}{q - \overline{q}}, \qquad \text{observe that} \qquad \lim_{q \to 1} [X]_q = X. \tag{18}$$

Note that in (17), Zhang [27] replaces $(-)^{[a]} E^a{}_a - (-)^{[a+1]} E^{a+1}{}_{a+1}$ with the more convenient expression h_a.

In fact, (16) generalises to a more useful result (proven in [8]):

$$[E^a{}_b, E^b{}_a] = \frac{K_a \overline{K}_b - \overline{K}_a K_b}{q_a - \overline{q}_a} = \overline{\Delta}_a(K_a \overline{K}_b - \overline{K}_a K_b), \qquad (19)$$

viz:

$$[E^a{}_b, E^b{}_a] = (-)^{[a]} \left[(-)^{[a]} E^a{}_a - (-)^{[b]} E^b{}_b \right]_q.$$

We also have, for $|a - b| > 1$, the commutations:

$$E^{a+1}{}_a E^{b+1}{}_b = E^{b+1}{}_b E^{a+1}{}_a \quad \text{and} \quad E^a{}_{a+1} E^b{}_{b+1} = E^b{}_{b+1} E^a{}_{a+1}. \qquad (20)$$

4. The squares of the odd simple generators are zero:

$$(E^m{}_{m+1})^2 = (E^{m+1}{}_m)^2 = 0.$$

In fact, we may show that this implies that the squares of *nonsimple* odd generators are also zero:

$$(E^a{}_b)^2 = 0, \qquad [a] \neq [b]. \qquad (21)$$

5. Lastly, we have the $U_q[gl(m|n)]$ *Serre relations*; their inclusion ensures that the algebra is reduced enough to be *simple*. For $a \neq m$:

$$(E^{a+1}{}_a)^2 E^{a\pm1+1}{}_{a\pm1} - \nabla E^{a+1}{}_a E^{a\pm1+1}{}_{a\pm1} E^{a+1}{}_a + E^{a\pm1+1}{}_{a\pm1} (E^{a+1}{}_a)^2 = 0$$
$$(E^a{}_{a+1})^2 E^{a\pm1}{}_{a\pm1+1} - \nabla E^a{}_{a+1} E^{a\pm1}{}_{a\pm1+1} E^a{}_{a+1} + E^{a\pm1}{}_{a\pm1+1} (E^a{}_{a+1})^2 = 0.$$

where to save space, we have introduced the notation: $\nabla \triangleq q + \overline{q}$. These may be more succinctly expressed using nonsimple generators. Noting that for $a \neq m$, we have $q_a = q_{a+1}$, the above become, again for $a \neq m$:

$$
\left.
\begin{array}{rrcl}
\text{(a)} & E^{a+1}{}_a E^{a+2}{}_a &=& q_a E^{a+2}{}_a E^{a+1}{}_a \\
\text{(b)} & E^a{}_{a+1} E^a{}_{a+2} &=& q_a E^a{}_{a+2} E^a{}_{a+1} \\
\text{(c)} & E^{a+1}{}_{a-1} E^{a+1}{}_a &=& q_a E^{a+1}{}_a E^{a+1}{}_{a-1} \\
\text{(d)} & E^{a-1}{}_{a+1} E^a{}_{a+1} &=& q_a E^a{}_{a+1} E^{a-1}{}_{a+1}.
\end{array}
\right\} \quad (22)
$$

10

Alternatively, if we define a *q graded commutator*:

$$[X, Y]_q \triangleq XY - (-)^{[X][Y]} qYX,$$

and, equivalently, a $\overline{q}$ *graded commutator* by replacing q with $\overline{q}$, then (22) may be more elegantly expressed:

$$[X, [X, Y]_q]_{\overline{q}} = 0, \tag{23}$$

where the pair (X, Y) represents the four pairs $(E^a{}_{a+1}, E^{a\pm1}{}_{a\pm1+1})$ and $(E^{a+1}{}_a, E^{a\pm1+1}{}_{a\pm1})$. Equivalently, we may exchange q and $\overline{q}$ in (23). Note that in these cases, the parity factors $(-)^{[X][Y]}$ in (23) are always $+1$ as $a \neq m$ in $E^a{}_{a+1}$ and $E^{a+1}{}_a$, viz the graded commutators degenerate to ordinary commutators.

The $a \neq m$ Serre relations are complemented by a pair dealing with the case $a = m$:

$$
\begin{aligned}
E^{m+1}{}_m E^{m+2}{}_{m-1} &= -E^{m+2}{}_{m-1} E^{m+1}{}_m \\
E^m{}_{m+1} E^{m-1}{}_{m+2} &= -E^{m-1}{}_{m+2} E^m{}_{m+1},
\end{aligned}
$$

more succinctly expressed as:

$$\left[E^{m+1}{}_m, E^{m+2}{}_{m-1} \right] = \left[E^m{}_{m+1}, E^{m-1}{}_{m+2} \right] = 0,$$

where, as the generators are all odd, the graded commutators are read as anticommutators.

Observe that if either m or n is 1, there are actually *no* $U_q[gl(m|n)]$ Serre relations; making life a little simpler.

These relations tell us, in principle, how to reexpress products of simple generators. In general, to reexpress a product containing *non*simple generators, those nonsimple generators must first be recursively expanded using (8a), with any graded commutators expanded by linearity, before the above relations can be invoked.

The above description of the $U_q[gl(m|n)]$ relations should convince the gentle reader that $U_q[gl(m|n)]$ has a formidable structure. To facilitate examination of its representation theory, in §4, we will rewrite the $U_q[gl(m|n)]$ relations into a PBW basis formulation, which is suitable for implementation on a computer.

2.3 $U_q[gl(m|n)]$ root system

We next introduce the $U_q[gl(m|n)]$ root system, which is identical to that of $gl(m|n)$. We will have use for it in §3.2, §4.1 and §5.1.

Where the $gl(m|n)$ Cartan subalgebra is denoted by H, its dual, the $gl(m|n)$ weight space H^*, has a basis given by the $gl(m|n)$ *fundamental weights* $\{\varepsilon_a\}_{a \in \mathcal{I}}$, which are lists of zeros of length $m + n$, with a 1 in position a. The ε_a inherit a grading from that on the indices. As H and H^* are dual, where $\{e^b{}_b\}_{b \in \mathcal{I}}$ are the $gl(m|n)$ Cartan generators, we have the form: $e^b{}_b(\varepsilon_a) \triangleq \delta_{ab}$. On H^*, we have an invariant symmetric bilinear form $(\cdot, \cdot) : H^* \times H^* \to \mathbb{C}$, defined by:

$$(\varepsilon_a, \varepsilon_b) \triangleq (-)^{[a]} \delta_{ab},$$

and extended by linearity.

Next, to each non-Cartan $gl(m|n)$ generator $e^a{}_b$, there corresponds a $gl(m|n)$ *root* $\alpha_{ab} \triangleq \varepsilon_a - \varepsilon_b$, which is the weight of $e^a{}_b$ in the adjoint representation.[3] For our purposes, it is convenient to bastardise the notation. Also permitting $a = b$, we will refer to α_{ab} as the 'weight' of $e^a{}_b$:

$$\text{wt}(e^a{}_b) \triangleq \varepsilon_a - \varepsilon_b = \alpha_{ab}, \tag{24}$$

indicating that within a $gl(m|n)$ weight module (see §3), the action of a generator $e^a{}_b$ sends a vector of weight γ to another of weight $\gamma + \alpha_{ab}$.

The roots inherit a grading from the indices: $[\alpha_{ab}] \equiv [e^a{}_b]$. Further, we assign signs to them in accordance with those of these generators, viz that if $e^a{}_b$ is a lowering generator, the corresponding root α_{ab} is said to be *negative*, written $\alpha_{ab} \prec 0$, and if $e^a{}_b$ is a raising generator, then α_{ab} is said to be *positive*, written $\alpha_{ab} \succ 0$. To illustrate, weights for the $U_q[gl(3|1)]$ lowering generators are presented in Table 1.

X	$\text{wt}(X)$	
$E^4{}_3$	$(\ \ 0,\ \ \ 0, -1 \mid +1)$	
$E^4{}_2$	$(\ \ 0, -1,\ \ \ 0 \mid +1)$	odd
$E^4{}_1$	$(-1,\ \ \ 0,\ \ \ 0 \mid +1)$	
$E^3{}_2$	$(\ \ 0, -1, +1 \mid\ \ \ 0)$	
$E^3{}_1$	$(-1,\ \ \ 0, +1 \mid\ \ \ 0)$	even
$E^2{}_1$	$(-1, +1,\ \ \ 0 \mid\ \ \ 0)$	

Table 1: Weights (all negative) of the $U_q[gl(3|1)]$ lowering generators.

Using this notation, $gl(m|n)$ has the following *simple*, positive roots:

$$\alpha_{a,a+1} \triangleq \varepsilon_a - \varepsilon_{a+1}, \qquad a = 1, \ldots, m + n - 1.$$

Apart from the single odd simple root $\alpha_{m,m+1}$, the simple positive roots are all even. (Of various choices for Lie superalgebra root systems, this *distinguished root system* is unique in containing only one odd simple root.)

---

[3]We apologise for the overloading of α. The notation in this subsection will go no further.

Then, we define Δ_i^+ to be the set of $gl(m|n)$ positive roots of grading i, and Δ^+ to be the union of the Δ_i^+, viz:[4]

$$\Delta_i^+ = \{\gamma : [\gamma] = i \text{ and } \gamma \succ 0\}, \qquad\qquad \Delta^+ = \Delta_0^+ \cup \Delta_1^+,$$

where $[\gamma]$ denotes the grading of the root γ.

In terms of these, we define the half sums of all positive even and odd $gl(m|n)$ roots, and their graded sum ρ:

$$\rho_i = \tfrac{1}{2} \sum_{\gamma \in \Delta_i^+} \gamma, \qquad\qquad \rho = \tfrac{1}{2} \sum_{\gamma \in \Delta^+} (-)^{[\gamma]} \gamma = \sum_{i=0,1} (-)^i \rho_i,$$

viz $\rho = \rho_0 - \rho_1$. Specifically, for $gl(m|n)$ (and hence for $U_q[gl(m|n)]$), we find [11, p6207]:[5]

$$
\begin{aligned}
\rho_0 &= \tfrac{1}{2} \sum_{a=1}^{m} (m - 2a + 1)\,\varepsilon_a + \tfrac{1}{2} \sum_{a=m+1}^{m+n} (m + n - 2a + 1)\,\varepsilon_a \\
\rho_1 &= \tfrac{1}{2} \sum_{a=1}^{m} n\varepsilon_a - \tfrac{1}{2} \sum_{a=m+1}^{m+n} m\varepsilon_a \\
\rho &= \tfrac{1}{2} \sum_{a=1}^{m} (m - n - 2a + 1)\,\varepsilon_a + \tfrac{1}{2} \sum_{a=m+1}^{m+n} (2m + n - 2a + 1)\,\varepsilon_a.
\end{aligned}
$$

2.4 $U_q[gl(m|n)]$ as a Hopf superalgebra

$U_q[gl(m|n)]$ may be regarded as a Hopf superalgebra when equipped with the following (compatible!) coproduct Δ, counit ε and antipode S structures. The material is taken from [28, p1238], except that we have modified the definition of the coproduct and antipode so that they have increased symmetry. This material is included for completeness; in §6, we will only have need for the coproduct. $U_q[gl(m|n)]$ is in fact a *quasitriangular Hopf superalgebra*, i.e. it possesses a (universal) R matrix.

We first introduce some notation. A homomorphism or antihomomorphism H on $U_q[gl(m|n)]$ is described as *($\mathbb{Z}_2$) graded* if it is compatible with the graded commutator, viz:

$$H([X,Y]) = [H(X), H(Y)],$$

where the latter graded commutator may even exist on an ungraded space, e.g. $\mathbb{C}$, where it is actually trivial.

This means that a graded homomorphism H and a graded antihomomorphism A must necessarily satisfy:

$$H(XY) = H(X)H(Y) \qquad \text{and} \qquad A(XY) = (-)^{[X][Y]} A(Y)A(X),$$

of which only the latter varies from the usual, ungraded situation.

---

[4]We apologise for the overloading of Δ. In practice, this Δ will only appear with a positive superscript, so it is easily distinguishable.

[5]We make a correction to [11], which appears to cite an error reproduced several times before and after, e.g. it appears in Zhang [28]. To whit, the term "$2m$" in the formula for ρ repeatedly appears as "m".

2.4.1 Coproduct Δ

The coproduct (a.k.a. comultiplication) is a $\mathbb{Z}_2$ graded algebra homomorphism $\Delta : U_q[gl(m|n)] \to U_q[gl(m|n)] \otimes U_q[gl(m|n)]$, defined by:

$$
\left.
\begin{array}{llll}
\text{(a)} & \Delta(E^{a+1}{}_a) & = & E^{a+1}{}_a \otimes \overline{K}_a^{\frac{1}{2}} K_{a+1}^{\frac{1}{2}} + K_a^{\frac{1}{2}} \overline{K}_{a+1}^{\frac{1}{2}} \otimes E^{a+1}{}_a \\[2mm]
\text{(b)} & \Delta(E^a{}_{a+1}) & = & E^a{}_{a+1} \otimes \overline{K}_a^{\frac{1}{2}} K_{a+1}^{\frac{1}{2}} + K_a^{\frac{1}{2}} \overline{K}_{a+1}^{\frac{1}{2}} \otimes E^a{}_{a+1} \\[2mm]
\text{(c)} & \Delta(K_a) & = & K_a \otimes K_a,
\end{array}
\right\} \tag{25}
$$

and extended to all of $U_q[gl(m|n)]$ by:

$$
\Delta(XY) = \Delta(X)\Delta(Y), \qquad \text{for all } X, Y \in U_q[gl(m|n)]. \tag{26}
$$

Observe that Δ preserves grading, viz that $[\Delta(X)] = [X]$ for homogeneous $X \in U_q[gl(m|n)]$, where we have $[X \otimes Y] \triangleq [X] + [Y]$.

By substitution of q^N for q in (25c), we discover:

$$
\Delta(K_a^N) = K_a^N \otimes K_a^N,
$$

and, setting $N = 0$, hence $\Delta(\text{Id}) = \text{Id} \otimes \text{Id}$, as expected.

Before proceeding, we mention that our definition in (25) is only one of various possibilities; we have chosen it for its symmetry. In fact, in comparison with the literature, our Δ agrees with that of [13], and differs from that of Zhang [28] and that of my PhD thesis [4]. We mention that given *any* coproduct, it is possible to write down another coproduct structure, the "opposite coproduct": $\Delta^T \triangleq T \cdot \Delta$. Here, the *twist map* T is an operator on the tensor product $U_q[gl(m|n)] \otimes U_q[gl(m|n)]$, defined for homogeneous $X, Y \in U_q[gl(m|n)]$ by:

$$
T(X \otimes Y) = (-)^{[X][Y]}(Y \otimes X).
$$

More relevant to our purposes here, we may extend the expression for the coproduct for simple generators to that for nonsimple generators. Firstly, as in (8), writing $S_b^a \triangleq \text{sign}(a - b)$, we may cheerfully rewrite (25) for the simple generators $E^a{}_b$, for any $|a - b| = 1$:

$$
\Delta(E^a{}_b) = E^a{}_b \otimes K_a^{\frac{1}{2}S_b^a} \overline{K}_b^{\frac{1}{2}S_b^a} + \overline{K}_a^{\frac{1}{2}S_b^a} K_b^{\frac{1}{2}S_b^a} \otimes E^a{}_b. \tag{27}
$$

Using this notation, we prove in Lemma 3 in Appendix A.2 the following more general statement, for *any* valid indices a, b:

$$
\begin{aligned}
\Delta(E^a{}_b) = {} & E^a{}_b \otimes K_a^{\frac{1}{2}S_b^a} \overline{K}_b^{\frac{1}{2}S_b^a} + \overline{K}_a^{\frac{1}{2}S_b^a} K_b^{\frac{1}{2}S_b^a} \otimes E^a{}_b - {} \\
& S_b^a \sum_c \Delta_c \left(\overline{K}_c^{\frac{1}{2}S_b^a} K_b^{\frac{1}{2}S_b^a} E^a{}_c \otimes E^c{}_b K_a^{\frac{1}{2}S_b^a} \overline{K}_c^{\frac{1}{2}S_b^a} \right),
\end{aligned} \tag{28}
$$

where the sum ranges over all c strictly between a and b, and is simply ignored if $|a - b| \leqslant 1$. Where $a = b$, the statement is also true; this is made clear when the equivalence $K_a = q^{(-)^{[a]}E^a{}_a}$ is noted.

Lastly, we apologise for even further overloading the definition of Δ; to be sure, the coproduct will only appear with parentheses enclosing its argument.

2.4.2 Counit ε

The counit $\varepsilon : U_q[gl(m|n)] \rightarrow \mathbb{C}$, is also a $\mathbb{Z}_2$ graded algebra homomorphism, defined by:

$$\varepsilon(E^{a\pm1}{}_a) = \varepsilon(E^a{}_{a\pm1}) = 0, \qquad \varepsilon(K_a) = 1,$$

and extended to all of $U_q[gl(m|n)]$ by $\varepsilon(XY) = \varepsilon(X)\varepsilon(Y)$. Again, we have $\varepsilon(K_a^N) = 1$, and, setting $N = 0$, thus $\varepsilon(\mathrm{Id}) = 1$, as expected. We apologise for overloading the definition of ε as the counit with the $gl(m|n)$ fundamental weights (see §2.3). As we shall have no further use for the counit, we are safe.

2.4.3 Antipode S

Lastly, the antipode $S : U_q[gl(m|n)] \rightarrow U_q[gl(m|n)]$, is a $\mathbb{Z}_2$ graded algebra *antiauto*morphism, defined by:

$$
\begin{aligned}
S(E^{a+1}{}_a) &= -K_a^{\frac{1}{2}} K_{a+1}^{\frac{1}{2}} E^{a+1}{}_a \\
S(E^a{}_{a+1}) &= -\overline{K}_a^{\frac{1}{2}} \overline{K}_{a+1}^{\frac{1}{2}} E^a{}_{a+1} \\
S(K_a) &= \overline{K}_a,
\end{aligned}
$$

and extended to all of $U_q[gl(m|n)]$ by:

$$S(XY) = (-)^{[X][Y]} S(Y) S(X),$$

for homogeneous $X, Y \in U_q[gl(m|n)]$. Again, immediately $S(K_a^N) = \overline{K}_a^N$, and thus $S(\mathrm{Id}) = \mathrm{Id}$, as expected.

S is perhaps better expressed in terms of the notation introduced for the coproduct Δ. We have, for simple generators $E^a{}_b$, where $|a - b| = 1$:

$$S(E^a{}_b) = -K_a^{\frac{1}{2}S_b^a} K_b^{\frac{1}{2}S_b^a} E^a{}_b. \tag{29}$$

This result is also valid for the case $a = b$ (where $S_a^a = \mathrm{sign}(a-a) = 0$), and the formula degenerates to $S(E^a{}_a) = -K_a^0 K_a^0 E^a{}_a = -E^a{}_a$, which is equivalent to $S(K_a) = \overline{K}_a$. Furthermore, a direct inductive proof[6] shows that (29) generalises to the case of nonsimple generators $E^a{}_b$, so that (29) is, in a sense, the most general expression of S.

---

[6]An example proof, albeit for a different definition of S, is provided in Lemma 3 of [27].

3 Highest weight $U_q[gl(m|n)]$ representations π_λ

3.1 Introduction

The construction of highest weight representations for $U_q[gl(m|n)]$ involves initially postulating a *highest weight vector*, which we shall call $|1\rangle$. The action of the $U_q[gl(m|n)]$ Cartan generators on $|1\rangle$ is that of scalar multipliers; the details of these multiplications are encoded in the *weight* of $|1\rangle$. Thus, if we are dealing with a representation labeled:

$$\lambda \equiv (\lambda_1, \ldots, \lambda_m \mid \lambda_{m+1}, \ldots, \lambda_{m+n}) = \sum_{i=1}^{m+n} \lambda_i \varepsilon_i, \tag{30}$$

where the ε_i are $gl(m|n)$ fundamental weights (see §2.3), then we intend $|1\rangle$ to have weight λ, that is the action of K_a on $|1\rangle$ is:[7]

$$K_a \cdot |1\rangle \equiv \pi_\lambda(K_a) \cdot |1\rangle \triangleq q_a^{\lambda_a} |1\rangle. \tag{31}$$

Substituting q^N for q, we immediately have that:

$$K_a^N \cdot |1\rangle = q_a^{N\lambda_a} |1\rangle. \tag{32}$$

We implement the notion that $|1\rangle$ is a *highest* weight vector by declaring that it be annihilated by the actions of *all raising* generators:

$$E^a{}_b \cdot |1\rangle \triangleq 0, \qquad a < b. \tag{33}$$

The module V_λ is then defined by the action of all possible products of the $U_q[gl(m|n)]$ lowering generators on $|1\rangle$. We may determine a basis B_λ for V_λ, with elements $|i\rangle$ defined by:

$$|i\rangle \triangleq \beta_i\, X_{i_1} X_{i_2} \cdots X_{i_p} \cdot |1\rangle,$$

where the X_{i_j} are $U_q[gl(m|n)]$ generators, p is the number of generators in the product, and β_i is a normalisation constant.

We call B_λ a *graded weight* basis, meaning that we may assign to it (i.e. to V_λ) a grading consistent with that of $U_q[gl(m|n)]$, and a system of *weights* (both, see §5). For our specific choices of λ (again, see §5), V_λ is finite-dimensional.

3.2 The Kac induced module construction

The *Kac induced module construction* (KIMC) is a two-stage process which efficiently implements the construction of B_λ.

- Firstly, we construct a basis B_λ^0 for the so-called 'even subalgebra submodule' $V_\lambda^0 \subset V_\lambda$; this being the module of highest weight λ of the $U_q[gl(m|n)]$ 'even subalgebra' $U_q[gl(m) \oplus gl(n)]$, viz the algebra generated by the even generators of $U_q[gl(m|n)]$. That is, V_λ^0 is defined by the action of all possible combinations of *even lowering* generators on $|1\rangle$, where we have declared that $|1\rangle$ is annihilated by the action of all *even raising* generators.

---

[7] For consistency of the weight notation between $gl(m|n)$ and $U_q[gl(m|n)]$, (31) tells us that λ_a is actually the weight of $|1\rangle$ in terms of the $U_q[gl(m|n)]$ 'generators' $E^a{}_a$.

- Secondly, V_λ is induced from V_λ^0 by the repeated action of the odd lowering generators on V_λ^0, subject to the proviso that V_λ^0 is annihilated by the (unique) odd raising generator, to whit: $E^m{}_{m+1} \cdot V_\lambda^0 \triangleq \{0\}$. This implies that $E^a{}_b \cdot V_\lambda^0 = \{0\}$ for *all* odd raising generators $E^a{}_b$, $a < b$. Thus, we construct B_λ from B_λ^0.

  In this process, there is a subtlety: the resultant "Kac module" (i.e. V_λ) may not be irreducible. However, we shall choose λ such that π_λ is a so-called *typical* representation [17, 28], ensuring that V_λ *is* irreducible.

3.3 Dimension of V_λ

For arbitrary typical highest weight $U_q[gl(m|n)]$ representations V_λ, we have the following *Kac–Weyl dimension formula* [17]:

$$\dim(V_\lambda) = 2^{mn} \cdot \dim(V_\lambda^0), \qquad \text{where} \quad \dim(V_\lambda^0) = \prod_{\gamma \in \Delta_0^+} \frac{(\lambda + \rho_0, \gamma)}{(\rho_0, \gamma)}, \quad (34)$$

where Δ_0^+, ρ_0 and the inner product $(\cdot, \cdot)$ on the $gl(m|n)$ fundamental weights are presented in §2.3.

For the specific choice $\lambda = \Lambda = (\dot 0_m | \dot\alpha_n)$, for even positive roots γ, we have:

$$(\Lambda, \gamma) = \tfrac{1}{2} \sum_{a<b, [a]=[b]} (\Lambda, \varepsilon_a - \varepsilon_b) = \tfrac{1}{2} \sum_{a<b, [a]=[b]} (-)^{[a]}(\Lambda_a - \Lambda_b) = 0,$$

as $\Lambda_a = \Lambda_b$ for $[a] = [b]$, thus $\dim(V_\Lambda^0) = 1$, hence $\dim(V_\Lambda) = 2^{mn}$, which simplifies things. Details of the KIMC for this case are presented in §5.

3.4 Matrix elements

To construct explicit *matrix elements* $\pi_\Lambda(X)$ for a particular $U_q[gl(m|n)]$ generator X, the action of X on *each* of the basis vectors of B_λ must be determined.

Whilst the action of the generators on $|1\rangle$ is predefined, more generally, the determination of the action of X on an arbitrary vector $|i\rangle$ requires the rendering of a string of generators into a normal ordering and the application of the 'KIMC rules' to simplify that normally ordered expression into (a multiple of) a basis vector.

Thus, we must first determine an appropriate ordering (see §4.1), and then describe an appropriate set of generator commutations to implement that ordering (viz the PBW lemma of §4.2). By the latter, we mean that we intend not to use all the commutators of §2.2.5 directly. Instead, we shall use expressions taken from lemmas in [27, 28] for the commutations between non-Cartan generators.

With these tools, we proceed to build bases and explicit matrix elements for our particular representations in §5.

4 A normal ordering and a PBW lemma

Finding a normal ordering for a string of $U_q[gl(m|n)]$ algebra generators involves the recursive use of commutation relations to rewrite the string as a sum of strings, with respect to some chosen (hopefully natural) ordering. Both the initial string and the resultant may contain initial scalar multipliers, which for $U_q[gl(m|n)]$ are typically algebraic expressions in q. When we speak of the length of a string, we shall ignore these scalars.

A PBW lemma describes the appropriate commutations, but we must determine an ordering ourselves. Perhaps the most natural ordering is purely by weight (see (24) in §2.3), but there are reasons for choosing other orderings.

4.1 A normal ordering for $U_q[gl(m|n)]$

We begin with the convention that if generators G_1 and G_2 are ordered, viz $G_1 \leqslant G_2$, then the string $G_1 G_2$ is ordered. With this, the ordering we choose is based on the following principles:

1. Our string will often be regarded as (right) acting on the highest weight vector $|1\rangle \in V_\lambda$, and the KIMC directs us to first build an even subalgebra submodule V_λ^0 (see §3.2 and §5.1) based on this $|1\rangle$, i.e. to define basis vectors of V_λ^0 in terms of the right actions of strings of even lowering generators on $|1\rangle$. Thus, we require even generators to be greater than odd generators, i.e. within normally-ordered strings, even generators lie to the right of odd ones.

2. Within the even generators, $|1\rangle$ is always annihilated by the (right) action of raising generators, so these must be greatest, i.e. rightmost. By symmetry, we then demand that the least amongst the even generators are the lowering generators, so the Cartan must be lie between the even lowering and the even raising.

3. Within the odd generators, the (right) action of the raising generators always annihilates any vectors from V_λ^0, so the odd lowering must be lesser than the odd raising.

4. Within the five equivalence classes created by these considerations, non-Cartan generators are ordered by increasing weight. Doing this ensures that squares of odd generators can be systematically identified and annihilated; it also facilitates a systematic way of defining basis vectors for V_λ (see §5).

   Furthermore, (powers of) Cartan generators are ordered by index, viz $K_a^M \leqslant K_b^N$ if $a \leqslant b$. Doing this ensures that powers of the same generator may be combined.

We call this ordering "$OL < OR < EL < C < ER$". It differs slightly from that (implicitly) described in [28, p1240], viz $OR < OL < EL < C < ER$. To implement it, we say that distinct *weights* γ_1 and γ_2 are ordered (viz $\gamma_1 \prec \gamma_2$) if the first nonzero component of $\gamma_1 - \gamma_2$ is positive. Then, say that we are comparing generators G_1 and G_2; where γ_i is the weight of G_i (see (24) in §2.3); L_i is the 'lifting' of G_i, being -1, 0 or $+1$ if G_i is a lowering, Cartan or raising generator, respectively; and, if G_i is Cartan, then let a_i be its index (the exponent is unimportant). Then:

$$A \vee (B \wedge (C \vee (D \wedge ((E \wedge F) \vee G)))) \iff G_1 \leqslant G_2,$$

where:

A	is	$[G_1] > [G_2]$	G_1 is odd and G_2 even
B	is	$[G_1] = [G_2]$	both odd or both even
C	is	$L_1 < L_2$	ordered liftings
D	is	$L_1 = L_2$	same liftings
E	is	$L_1 = 0$	G_1 is Cartan
F	is	$a_1 \leqslant a_2$	(implicitly) both Cartan and ordered
G	is	$\gamma_1 \preccurlyeq \gamma_2$	ordered within (implicitly non-Cartan) class.

To illustrate the ordering, for $U_q[gl(3|1)]$, with reference to Table 1, we have:

$$\underbrace{E^4{}_3 < E^4{}_2 < E^4{}_1}_{\text{Odd Lowering}} < \underbrace{E^1{}_4 < E^2{}_4 < E^3{}_4}_{\text{Odd Raising}} <$$

$$\underbrace{E^3{}_2 < E^2{}_1 < E^3{}_1}_{\text{Even Lowering}} < \underbrace{K_1^{N_1} < K_2^{N_2} < K_3^{N_3}}_{\text{Cartan}} < \underbrace{E^1{}_3 < E^1{}_2 < E^2{}_3}_{\text{Even Raising}}.$$

This ordering ensures that in the KIMC, the action of a normally-ordered string of generators on a highest weight vector $|1\rangle$ may be evaluated by the following *ordered* steps.

1. If there are any terminal even raising generators, then the string evaluates to 0.

2. If there are terminal Cartan generators, then these may be replaced by their known scalar actions on $|1\rangle$, and the string is reduced in length.

3. Next, the action of any even lowering generators is considered. In the general situation, these map $|1\rangle$ to another basis vector of V_λ^0. For our particular modules V_Λ, where $\Lambda = (\dot{0}_m | \dot{\alpha}_n)$, as V_Λ^0 is one dimensional, $|1\rangle$ is in fact *annihilated* by the even lowering generators, so if there are *any* even raising generators, then the string evaluates to 0.

4. Next, the odd raising generators annihilate any vectors of V_λ^0, so if any are present, then the string evaluates to 0.

5. Lastly, when the string is reduced to (a scalar multiple of) the action of some unrepeated odd lowering generators on a V_λ^0 basis vector, that residual string may be identified as (a scalar multiple of) a particular V_λ basis vector.

We mention that although our PBW lemma provides us with the *means* to normally order generator strings, the normal ordering is a computationally expensive process. Firstly, *each* exchange may generate up to two extra terms in a sum,[8] so there is a geometric increase in the number of terms with exchanges. Secondly, implementation of the exchanges is really a sorting procedure, but we have not been able to implement an efficient algorithm – we in fact use the dumbest possible opportunistic exchange. This failure is partly due to the complexities in developing a sorting algorithm in the presence of the continual creation of extra terms.

Thus, the process to normally order generator strings requires time and storage which both of which grow at least exponentially with string length. Using MATHEMATICA, we currently get into serious trouble beyond length 8.

4.2   Commutations implementing the normal ordering

To implement the normal ordering described in §4.1, we describe here a set of generator-exchanging commutations. The material originates in [27, 28]; we have modified the results a little in light of (15), rearranged many things, and corrected several minor mistakes. In what follows, we intend distinct abstract indices to represent different concrete indices.

4.2.1   A PBW commutator lemma

The following result contains some corrections to the original [28]. In it, we use the notation presented in (10).

---

[8]Although the exchanges can in fact add up to two extra terms, in practice they add only one extra term, but they can be 'sum-neutral' or even subtract a term.

Lemma 1

We have the following commutations. Firstly, (16) generalises to the case of nonsimple generators (19), viz:

$$\left[E^a{}_b, E^b{}_a\right] = \overline{\Delta}_a(K_a\overline{K}_b - \overline{K}_aK_b) \qquad all\ a, b. \tag{35}$$

Secondly, where there are three distinct indices, we have:

$$\left[E^a{}_c, E^c{}_b\right] = \left\{ \begin{array}{lll} \text{(a)} & K_c\overline{K}_bE^a{}_b & c < b < a \\ \text{(b)} & E^a{}_bK_a\overline{K}_c & c < a < b \\ \text{(c)} & E^a{}_bK_c\overline{K}_a & b < a < c \\ \text{(d)} & K_b\overline{K}_cE^a{}_b & a < b < c \end{array} \right\} \tag{36}$$

$$\left[E^c{}_a, E^c{}_b\right] = \left[E^a{}_c, E^b{}_c\right] = 0 \qquad a < c < b \ \ or \ \ b < c < a \tag{37}$$

$$E^c{}_aE^c{}_b = \left\{ \begin{array}{lll} \text{(a)} & (-)^{[E^c{}_b]} & a < b < c \\ \text{(b)} & (-)^{[E^c{}_a]} & c < a < b \end{array} \right\} q_cE^c{}_bE^c{}_a \tag{38}$$

$$E^a{}_cE^b{}_c = \left\{ \begin{array}{lll} \text{(a)} & (-)^{[E^b{}_c]} & a < b < c \\ \text{(b)} & (-)^{[E^a{}_c]} & c < a < b \end{array} \right\} q_cE^b{}_cE^a{}_c. \tag{39}$$

Thirdly, we describe the situation where there are no common indices, and we have $a < b$ and $c < d$. Let $S(\mathsf{x}, \mathsf{y})$ denote the set of integers $\{\mathsf{x}, \mathsf{x} + 1, \ldots, \mathsf{y}\}$. Then, if $S(a, b)$ and $S(c, d)$ are either disjoint or one is wholly contained within the other, viz $a < c < d < b$, $a < b < c < d$, $c < a < b < d$ or $c < d < a < b$, we have a total of 16 cases:

$$\left[E^a{}_b, E^c{}_d\right] = \left[E^a{}_b, E^d{}_c\right] = \left[E^b{}_a, E^c{}_d\right] = \left[E^b{}_a, E^d{}_c\right] = 0. \tag{40}$$

More interestingly, if there is some other overlap between the sets $S(a, b)$ and $S(c, d)$, viz $a < c < b < d$ or $c < a < d < b$, then we have the 8 cases:

$$\left[E^a{}_b, E^c{}_d\right] = \left\{ \begin{array}{lll} \text{(a)} & +\Delta_b & a < c < b < d \\ \text{(b)} & -\Delta_d & c < a < d < b \end{array} \right\} E^a{}_dE^c{}_b \tag{41}$$

$$\left[E^b{}_a, E^d{}_c\right] = \left\{ \begin{array}{lll} \text{(a)} & +\Delta_b & a < c < b < d \\ \text{(b)} & -\Delta_d & c < a < d < b \end{array} \right\} E^d{}_aE^b{}_c \tag{42}$$

$$\left[E^a{}_b, E^d{}_c\right] = \left\{ \begin{array}{lll} \text{(a)} & -\Delta_b\overline{K}_bK_cE^a{}_cE^d{}_b & a < c < b < d \\ \text{(b)} & +\Delta_dE^d{}_bE^a{}_c\overline{K}_aK_d & c < a < d < b \end{array} \right\} \tag{43}$$

$$\left[E^b{}_a, E^c{}_d\right] = \left\{ \begin{array}{lll} \text{(a)} & -\Delta_cE^b{}_dE^c{}_a\overline{K}_cK_b & a < c < b < d \\ \text{(b)} & +\Delta_a\overline{K}_dK_aE^c{}_aE^b{}_d & c < a < d < b \end{array} \right\}. \tag{44}$$

Rearranging the indices in Lemma 1 gives us the following simplified results.

- The entirety of (36) may be summarised by:

$$[E^a{}_c, E^c{}_b] = \begin{cases} K_c \overline{K}_b E^a{}_b & c < b < a \\ E^a{}_b K_a \overline{K}_c & c < a < b \\ E^a{}_b K_c \overline{K}_a & b < a < c \\ K_b \overline{K}_c E^a{}_b & a < b < c. \end{cases}$$

- The entirety of (37) to (39) may be summarised by:

$$E^a{}_c E^b{}_c = \kappa E^b{}_c E^a{}_c \qquad \text{and} \qquad E^c{}_a E^c{}_b = \kappa E^c{}_b E^c{}_a,$$

where:

$$\kappa \triangleq \begin{cases} 1 & \text{if } z(a,b,c) = c \\ (-)^{[E^{z(a,b,c)}{}_c]} \overline{q}_c^{S^a_b} & \text{else,} \end{cases}$$

and $z(a, b, c)$ picks out the middle element of $\{a, b, c\}$. (The 1 factor follows as $[E^a{}_c][E^b{}_c] = 0$ for c between a and b.)

- The entirety of (40) to (44) may be summarised by:

$$[E^a{}_b, E^c{}_d] = \begin{cases} +\Delta_b E^a{}_d E^c{}_b & a < c < b < d \\ -\Delta_d E^a{}_d E^c{}_b & c < a < d < b \\ +\Delta_a E^c{}_b E^a{}_d & b < d < a < c \\ -\Delta_c E^c{}_b E^a{}_d & d < b < c < a \\ -\Delta_b \overline{K}_b K_d E^a{}_d E^c{}_b & a < d < b < c \\ +\Delta_c E^c{}_b E^a{}_d \overline{K}_a K_c & d < a < c < b \\ -\Delta_c E^a{}_d E^c{}_b \overline{K}_c K_a & b < c < a < d \\ +\Delta_b \overline{K}_d K_b E^c{}_b E^a{}_d & c < b < d < a \\ 0 & a \neq b \neq c \neq d \quad \text{else.} \end{cases}$$

From these, we deduce the following rules for exchanges:

From (12), replace $K_a^M K_b^N$ **with** $K_b^N K_a^M$.

    If also $a = b$, then replace it with K_a^{M+N}.

        If also $M + N = 0$ then replace it with Id.

From (15), replace $K_a^N E^b{}_c$ **with** $q_a^{N(\delta_b^a - \delta_c^a)} E^b{}_c K_a^N$, and

    replace $E^b{}_c K_a^N$ **with** $q_a^{-N(\delta_b^a - \delta_c^a)} K_a^N E^b{}_c$.

From (19), replace $E^a{}_b E^b{}_a$ **with** $(-)^{[E^a{}_b]} E^b{}_a E^a{}_b + \overline{\Delta}_a (K_a \overline{K}_b - \overline{K}_a K_b)$.

From (21), replace $E^a{}_b E^a{}_b$ **with** 0 if $[E^a{}_b] = 1$.

Replace $E^a{}_c E^c{}_b$ **with:**

$$
\begin{array}{lll}
& E^a{}_b \; + \; q_c^{S_b^a} E^c{}_b E^a{}_c & a < c < b, \quad b < c < a \\
(-)^{[E^c{}_b]} E^c{}_b E^a{}_c \; + \; & K_c \overline{K}_b E^a{}_b & c < b < a \\
(-)^{[E^a{}_c]} E^c{}_b E^a{}_c \; + \; & E^a{}_b K_a \overline{K}_c & c < a < b \\
(-)^{[E^a{}_c]} E^c{}_b E^a{}_c \; + \; & E^a{}_b K_c \overline{K}_a & b < a < c \\
(-)^{[E^c{}_b]} E^c{}_b E^a{}_c \; + \; & K_b \overline{K}_c E^a{}_b & a < b < c.
\end{array}
$$

Replace $E^c{}_b E^a{}_c$ **with:**

$$
\begin{array}{lll}
q_c^{S_a^b} \, (E^a{}_c E^c{}_b \; - \; E^a{}_b) & a < c < b, \quad b < c < a \\
(-)^{[E^c{}_b]} \, (E^a{}_c E^c{}_b \; - \; K_c \overline{K}_b E^a{}_b) & c < b < a \\
(-)^{[E^a{}_c]} \, (E^a{}_c E^c{}_b \; - \; E^a{}_b K_a \overline{K}_c) & c < a < b \\
(-)^{[E^a{}_c]} \, (E^a{}_c E^c{}_b \; - \; E^a{}_b K_c \overline{K}_a) & b < a < c \\
(-)^{[E^c{}_b]} \, (E^a{}_c E^c{}_b \; - \; K_b \overline{K}_c E^a{}_b) & a < b < c.
\end{array}
$$

Replace $E^c{}_a E^c{}_b$ **with** $\kappa E^c{}_b E^c{}_a$, and

    replace $E^a{}_c E^b{}_c$ **with** $\kappa E^b{}_c E^a{}_c$, where:

$$
\kappa \triangleq \begin{cases} 1 & \text{if } z(a,b,c) = c \\ (-)^{[E^{z(a,b,c)}{}_c]} \overline{q}_c^{S_b^a} & \text{else,} \end{cases}
$$

    and where $z(a,b,c)$ picks out the middle element of $\{a, b, c\}$.

Replace $E^a{}_b E^c{}_d$ **with** $(-)^{[E^a{}_b][E^c{}_d]} E^c{}_d E^a{}_b + T$, where:

$$
T = \begin{cases}
+\Delta_b E^a{}_d E^c{}_b & a < c < b < d \\
-\Delta_d E^a{}_d E^c{}_b & c < a < d < b \\
+\Delta_a E^c{}_b E^a{}_d & b < d < a < c \\
-\Delta_c E^c{}_b E^a{}_d & d < b < c < a \\
-\Delta_b \overline{K}_b K_d E^a{}_d E^c{}_b & a < d < b < c \\
+\Delta_c E^c{}_b E^a{}_d \overline{K}_a K_c & d < a < c < b \\
-\Delta_c E^a{}_d E^c{}_b \overline{K}_c K_a & b < c < a < d \\
+\Delta_b \overline{K}_d K_b E^c{}_b E^a{}_d & c < b < d < a \\
0 & a \neq b \neq c \neq d \text{ else.}
\end{cases}
$$

5 The $U_q[gl(m|n)]$ representations $(\dot{0}_m|\dot{\alpha}_n)$

Fixing m and n, in this section we describe the use of a version of the Kac induced module construction (KIMC, see §3.2) in the brute-force construction of the $U_q[gl(m|n)]$ representation $\Lambda \triangleq (\dot{0}_m|\dot{\alpha}_n)$.

Alternatively, we might have implemented the results presented in [22, 23], which describe the use of a Gel'fand–Tsetlin basis to explicitly construct the actions for *essentially typical* representations[9] (this class includes our representation, which is actually *typical*). We avoid those fine results because we wish our code to be more general, but we pay a price for this in the currency of computational expense.

Strictly, this material applies only to *generic* q, that is q not a root of unity (in which case the representation theory changes drastically). Also, our representations are *unitary* only under some constraints on α (viz that α is real and either $\alpha > n - 1$ or $\alpha < 1 - m$, see [11]), and so we shall implicitly select these. In the application of our results to the computation of link invariants [7], the representation of the braid generator based on our quantum R matrix $\check{R}^{m,n}$ again contains the variables q and α. However, it turns out that $\check{R}^{m,n}$ is actually a valid braid generator for *any* q and α.

5.1 An orthonormal basis B for $V \equiv V_\Lambda$

Recall from §3.2 and §3.3 that $V \equiv V_\Lambda$ is of dimension 2^{mn}, and may be equipped with a grading compatible with that of $U_q[gl(m|n)]$. It is known that V contains no *weight multiplicities* (that is, V contains no constant weight subspaces of dimension greater than 1), so that a weight basis for V will contain no distinct vectors of the *same* weight, and this makes our task a little simpler.

Here, we use a version of the KIMC to construct a weight basis $B = \{|i\rangle\}_{i=1}^{2^{mn}}$ for V. That is, the 2^{mn} basis vectors $|i\rangle$ are defined in terms of the actions of all 2^{mn} possible nonrepeated, ordered combinations of the $U_q[gl(m|n)]$ simple lowering generators (there are mn of them) on a postulated highest weight vector $|1\rangle$. This $|1\rangle$ is further defined to be of unit length and annihilated by *all* $U_q[gl(m|n)]$ raising generators. Each of the vectors defined in this manner will be orthogonal to all other such vectors, and it is a straightforward matter to select constants to orthonormalise them. The resulting B is thus a graded orthonormal weight basis for V. Using it, in §5.2, we construct matrix elements for the $U_q[gl(m|n)]$ generators. In each subsection, we shall illustrate our results using $U_q[gl(3|1)]$.

---

[9] It turns out that these results sometimes hold for other representations, when various limits are evaluated using L'Hôpital formulae.

5.1.1 Details of the KIMC

The KIMC firstly instructs us to construct a basis B^0 for V^0, the submodule of V determined by the action of the $U_q[gl(m|n)]$ even subalgebra on $|1\rangle$. For our choice of Λ, in fact V^0 has dimension 1 (see §3.3), hence $B^0 = \{|1\rangle\}$.

This means that $|1\rangle$ is annihilated not only by all *raising* generators, by also by all *even* lowering generators. More generally, for other representations, we have to work harder to construct a basis for V^0; that process has a similar appearance to the following.

Secondly, B is induced from B^0 by the actions of all possible products of *odd lowering* generators on B^0. Thus, we must consider the set of all possible products of odd lowering generators. The PBW lemma allows us to reduce this set to that of all possible *ordered* products, and the knowledge that the square of odd generators is zero allows us to reduce it to the set of all possible *nonrepeated* ordered products, a *finite* set, ensuring that B is finite. Recalling that $U_q[gl(m|n)]$ has mn (simple and nonsimple) odd lowering generators:

$$OLGS = \{E^a{}_b : a = m+1, \ldots, m+n, b = 1, \ldots, m\}, \tag{45}$$

thus V is spanned by a set of vectors obtained by the actions of all possible ordered products (that is, strings of length 0 to mn) of $OLGS$ generators on $|1\rangle$. Indeed, this is the source of the factor 2^{mn} in the dimension formula (34). As V is known to have no weight multiplicities, this spanning set is itself the desired weight basis B. The set of ordered products of generators may be obtained from the power set $\mathcal{P}(OLGS)$, by replacing its elements with respective products – MATHEMATICA is well-suited to this.

To illustrate, for the $U_q[gl(3|1)]$ case, we have $OLGS = \{E^4{}_3, E^4{}_2, E^4{}_1\}$, and V has the following basis B:

$$\begin{pmatrix} |1\rangle \\ |2\rangle \\ |3\rangle \\ |4\rangle \\ |5\rangle \\ |6\rangle \\ |7\rangle \\ |8\rangle \end{pmatrix} = \begin{pmatrix} \beta_1 \mathrm{Id} \\ \beta_2 E^4{}_3 \\ \beta_3 E^4{}_2 \\ \beta_4 E^4{}_1 \\ \beta_5 E^4{}_3 E^4{}_2 \\ \beta_6 E^4{}_3 E^4{}_1 \\ \beta_7 E^4{}_2 E^4{}_1 \\ \beta_8 E^4{}_3 E^4{}_2 E^4{}_1 \end{pmatrix} \cdot |1\rangle, \tag{46}$$

where the β_i are scaling factors, which we shall select to normalise the $|i\rangle$; the redundant β_1 is implicitly 1.

5.1.2 Weights, gradings and an ordering for B

Our B is a *weight* basis, in that each of the $|i\rangle$ is of a definite *weight*. If we have, for some i:

$$|i\rangle = \beta_i E^{a_1}{}_{b_1} E^{a_2}{}_{b_2} \cdots E^{a_p}{}_{b_p} \cdot |1\rangle, \qquad (47)$$

for some $p \leqslant mn$, then we may define:

$$\mathrm{wt}(|i\rangle) = \Lambda + \sum_{j=1}^{p} \mathrm{wt}(E^{a_j}{}_{b_j}),$$

where $\mathrm{wt}(E^a{}_b)$ is the weight of the generator $E^a{}_b$ (see (24) in §2.3). As we defined $|1\rangle$ to be a highest weight vector, clearly we intend $\mathrm{wt}(|1\rangle) = \Lambda$.

Further, our B is $\mathbb{Z}_2$ *graded*, in that its elements are formed from the actions of products of graded $U_q[gl(m|n)]$ generators on the (zero) graded $|1\rangle$. The $\mathbb{Z}_2$ grading of $|i\rangle$ (as defined in (47)) is defined by:

$$[|1\rangle] \triangleq 0, \qquad [|i\rangle] \triangleq \sum_{j=1}^{p} [E^{a_j}{}_{b_j}] = p \pmod{2},$$

where the latter result holds as the $E^{a_j}{}_{b_j}$ are all odd. This $\mathbb{Z}_2$ grading on V is compatible with a notion of $\mathbb{Z}$ *graded level*, this being p, the number of factors in the product forming $|i\rangle$. This notion is relevant in the calculation – we recursively form the basis vectors in level l by the action of the $OLGS$ on the basis vectors of level $l-1$, for $l = 1, \ldots, mn$. We number our vectors $|i\rangle$ by decreasing weight within increasing $\mathbb{Z}$ graded levels. This ordering is important in that it simplifies the process of identifying an arbitrary string acting on $|1\rangle$, which is required in §5.2.

The weights and gradings of the basis vectors for our $U_q[gl(3|1)]$ example are supplied in Table 2 (cf. Table 1).

i	$\mathrm{wt}(|i\rangle)$				$[|i\rangle]$	$[|i\rangle]_{\mathbb{Z}}$
1	(0,	0,	0	α)	0	0
2	(0,	0,	-1	$\alpha+1$)	1	
3	(0,	-1,	0	$\alpha+1$)	1	1
4	(-1,	0,	0	$\alpha+1$)	1	
5	(0,	-1,	-1	$\alpha+2$)	0	
6	(-1,	0,	-1	$\alpha+2$)	0	2
7	(-1,	-1,	0	$\alpha+2$)	0	
8	(-1,	-1,	-1	$\alpha+3$)	1	3

Table 2: Weights and gradings of the basis vectors $|i\rangle$ of B, for $U_q[gl(3|1)]$, ordered by decreasing weight within increasing $\mathbb{Z}$ graded levels $[|i\rangle]_{\mathbb{Z}}$.

5.1.3 Normalisation of B

To investigate questions of orthogonality, we require an inner product on V. To whit, we introduce a basis $B^* = \{\langle i|\}_{i=1}^{2^{mn}}$ of V^* (the dual of V), by:

$$\langle i| \triangleq \beta_i^* \langle 1| \cdot \mathbf{E}^{b_p}{}_{a_p} \cdots \mathbf{E}^{b_2}{}_{a_2} \mathbf{E}^{b_1}{}_{a_1}, \tag{48}$$

where β_i^* is the complex conjugate of β_i, and $|i\rangle$ is as supplied in (47). Note that here, we explicitly intend the Hermitian conjugates $\mathbf{E}^b{}_a$ (see (8)), and not $E^b{}_a$. This ensures that B^* is conjugate to B, viz we have $|i\rangle^\dagger = \langle i|$, and that conjugate generators and conjugate basis vectors remain conjugates in their matrix representations. We assign weights and $\mathbb{Z}_2$ (and also $\mathbb{Z}$) gradings to the $\langle i|$ such that $\mathrm{wt}(\langle i|) \triangleq \mathrm{wt}(|i\rangle)$ and $[\langle i|] \triangleq [|i\rangle]$.

Using this conjugate basis, we define an inner product on V:

$$(|i\rangle, |j\rangle) \triangleq \langle i| \cdot |j\rangle \equiv \langle i|j\rangle, \tag{49}$$

where we implicitly have $\langle 1|1\rangle = 1$. Next, where $E^b{}_a$ is any raising generator (viz $b < a$), taking the conjugate of $E^b{}_a \cdot |1\rangle = 0$ yields $\langle 1| \cdot \mathbf{E}^a{}_b = 0$. Expanding the $\mathbf{E}^a{}_b$ into simple generators using (8b) shows that $\langle 1| \cdot E^{c+1}{}_c = 0$, for all *simple* lowering generators $E^{c+1}{}_c$, which in turn, recursively, yields $\langle 1| \cdot E^c{}_d = 0$ for *all* lowering generators $E^c{}_d$ with $c > d$. In sum, the equivalent of (33) is:

$$\langle 1| \cdot E^a{}_b = 0, \qquad a > b. \tag{50}$$

For completeness, we mention the (left) action of Cartan generators on $\langle 1|$, obtained by conjugating (32):

$$\langle 1| \cdot K_a^N = q_a^{N\lambda_a} \langle 1|. \tag{51}$$

More generally, the value of an inner product $\langle i|j\rangle$ may be calculated by the following procedure.

1. We substitute the definitions of $\langle i|$ and $|j\rangle$ (viz the appropriate versions of (48) and (47)) into (49), yielding a form $\beta_i^* \beta_j \langle 1| \cdot Z \cdot |1\rangle$, where Z is a string of generators. For our representations π_Λ, as V^0 is one-dimensional, Z will contain no even generators.

2. We use the PBW lemma to normally order Z. The resulting strings have their raising generators annihilating $|1\rangle$ pushed to their right hand ends and lowering generators annihilating $\langle 1|$ pushed to their left hand ends.

3. Implementing those annihilations, and evaluating the residual (scalar) Cartan generator actions on $|1\rangle$ and $\langle 1|$ (viz (32) and (51)), we convert the resulting expression to a scalar.

It turns out that vectors $|i\rangle$ and $|j\rangle$ with distinct weights satisfy $\langle i|j\rangle = 0$, viz $\langle i|j\rangle = \delta^i_j \langle i|i\rangle$. Recall that our $(\dot{0}_m|\dot{\alpha}_n)$ representations have no *weight multiplicities*, thus distinct basis vectors have distinct weights, hence our basis B is orthogonal. To make it ortho*normal*, we must select the β_i appropriately. This means that for each i, we must ensure that:

$$\langle i|i\rangle = \beta_i^* \beta_i \, \langle 1| \cdot \mathbf{E}^{b_p}{}_{a_p} \cdots \mathbf{E}^{b_2}{}_{a_2} \mathbf{E}^{b_1}{}_{a_1} \cdot E^{a_1}{}_{b_1} E^{a_2}{}_{b_2} \cdots E^{a_p}{}_{b_p} \cdot |1\rangle = 1.$$

Thus, for each i, we must use the commutations of the PBW lemma to normal order the following string Z_i:

$$Z_i = \mathbf{E}^{b_p}{}_{a_p} \cdots \mathbf{E}^{b_2}{}_{a_2} \mathbf{E}^{b_1}{}_{a_1} \cdot E^{a_1}{}_{b_1} E^{a_2}{}_{b_2} \cdots E^{a_p}{}_{b_p},$$

into $\mathrm{NO}(Z_i)$, and then apply the algebra-module actions (32), (33), (50), (51) and $\langle 1|1\rangle = 1$ to $\langle 1| \cdot \mathrm{NO}(Z_i) \cdot |1\rangle$ to yield, up to an arbitrary complex constant (phase factor), $\beta_i = (\langle 1| \cdot \mathrm{NO}(Z_i) \cdot |1\rangle)^{-1/2}$. The phase factor is unimportant; different choices simply lead to bases related by orthogonal transformations, and this will not affect our R matrices. In practice, we let the internal machinery of MATHEMATICA decide on phase factors for us – a human calculator might make more elegant choices.

So, at this stage, we have determined the constants β_i such that we have an orthonormal basis B for V. In general, for arbitrary representations of $U_q[gl(m|n)]$, these constants will be algebraic functions of the complex variable q. For our particular Λ, these functions will also contain the variable α.

To illustrate, for the $U_q[gl(3|1)]$ case, we have:

$$\left. \begin{aligned}
\beta_2 &= \quad [\alpha]_q^{-\frac{1}{2}} \\
\beta_3 &= \overline{q} \; [\alpha]_q^{-\frac{1}{2}} \\
\beta_4 &= \overline{q}^2 \, [\alpha]_q^{-\frac{1}{2}} \\
\beta_5 &= \overline{q} \; [\alpha]_q^{-\frac{1}{2}} [\alpha+1]_q^{-\frac{1}{2}} \\
\beta_6 &= \overline{q}^2 \, [\alpha]_q^{-\frac{1}{2}} [\alpha+1]_q^{-\frac{1}{2}} \\
\beta_7 &= \overline{q}^3 \, [\alpha]_q^{-\frac{1}{2}} [\alpha+1]_q^{-\frac{1}{2}} \\
\beta_8 &= \overline{q}^3 \, [\alpha]_q^{-\frac{1}{2}} [\alpha+1]_q^{-\frac{1}{2}} [\alpha+2]_q^{-\frac{1}{2}}.
\end{aligned} \right\} \tag{52}$$

where we have use the q-bracket (see (18)) to simplify the expressions. Thus, for example,

$$\beta_8 = \left(\frac{q^6 (q^\alpha - \overline{q}^\alpha)(q^{\alpha+1} - \overline{q}^{\alpha+1})(q^{\alpha+2} - \overline{q}^{\alpha+2})}{(q - \overline{q})^3} \right)^{-\frac{1}{2}}.$$

Our use of the q-bracket notation is more than cosmetic; the $U_q[gl(m|n)]$ symmetries manifest themselves *naturally* in π_Λ in these patterns, and if we do not recognise and incorporate them into our notation, then expressions rapidly become unreadable, and then intractable. Below, q-brackets will appear at every point, and even in our R matrices.

As mentioned in §4.1, the normal ordering of generator strings is a computationally expensive task. Here, the normal ordering of $Z_{2^{mn}}$ typically dominates the computations as it demands that we process a seriously disordered string of length $2mn$. A theoretical insight would be valuable here – for example an explicit formula for the normal ordering of arbitrary $\mathbf{E}^b{}_a E^a{}_b$ would help speed the evaluation of the β_i. The regularities apparent in the above example suggest such the existence of such a result, and (9) may also be of use. Alternatively, a more efficient computation of the β_i should be possible by the efficient reuse of previous calculations.

5.2 Matrix elements for $\pi \equiv \pi_\Lambda$

Having established an orthonormal basis B for the module $V \equiv V_\Lambda$ corresponding to the representation $\pi \equiv \pi_\Lambda$, we now use it to construct matrix elements $\pi(X)$ for $U_q[gl(m|n)]$ generators X.

Where $I_{2^{mn}}$ is the identity transformation on V, in the basis B we have the identity: $\sum_{i=1}^{2^{mn}} |i\rangle\langle i| = I_{2^{mn}}$, thus:

$$\pi(X) = \pi(X) \cdot \sum_{i=1}^{2^{mn}} |i\rangle\langle i| = \sum_{i=1}^{2^{mn}} (\pi(X) \cdot |i\rangle) \cdot \langle i| = \sum_{i=1}^{2^{mn}} (X \cdot |i\rangle) \cdot \langle i|, \qquad (53)$$

and the action $X \cdot |i\rangle$ may be computed knowing the expansion of $|i\rangle$ in terms of generator products and $|1\rangle$. Thus, if, as in (47), we have:

$$|i\rangle = \beta_i E^{a_1}{}_{b_1} E^{a_2}{}_{b_2} \cdots E^{a_p}{}_{b_p} \cdot |1\rangle,$$

then we may compute $X \cdot |i\rangle$ by the following process:

1. We again use the PBW basis commutators to normally order the string $Y \triangleq X E^{a_1}{}_{b_1} E^{a_2}{}_{b_2} \cdots E^{a_p}{}_{b_p}$, denoting the result by $\mathrm{NO}(Y)$.

2. We use the known actions of the raising and Cartan generators on $|1\rangle$ to reduce $\mathrm{NO}(Y) \cdot |1\rangle$ to an expression which is generally a *sum* of scalar-multiplied, normally ordered products of (odd) lowering generators acting on $|1\rangle$.

3. Identifying the terms in the resulting products as scalar multiples of various $|j\rangle$, we obtain the result. To whit, if $\mathrm{NO}(Y)$ contains a term of the form:

$$T \cdot |1\rangle = \theta E^{c_1}{}_{d_1} E^{c_2}{}_{d_2} \cdots E^{c_r}{}_{d_r} \cdot |1\rangle,$$

for some scalar θ, and r odd lowering generators $E^{c_j}{}_{d_j}$, and we know that:

$$|k\rangle = \beta_k E^{c_1}{}_{d_1} E^{c_2}{}_{d_2} \cdots E^{c_r}{}_{d_r} \cdot |1\rangle,$$

then we may replace $T \cdot |1\rangle$ with $\theta \overline{\beta}_k |k\rangle$.

Repeating this procedure for all 2^{mn} basis vectors $|i\rangle$, and substituting the results into (53) yields the required matrix element $\pi(X)$. Again, as the ordering chosen for the basis vectors $|i\rangle$ is compatible with the ordering used in the PBW lemma, this process is robust.

We now divide the construction of matrix elements into two phases. Firstly, in a direct implementation of the above, we build matrix elements for the Cartan and simple raising generators. We illustrate this for the $U_q[gl(3|1)]$ case, for the generators K_4 (in §5.2.1) and $E^3{}_4$ (in §5.2.2).

Having done that, in §5.2.3, we describe the construction of the remaining matrix elements, as they may be efficiently computed in terms of those for the simple lowering generators.

Further illustrations are provided in my PhD thesis [4], using the $U_q[gl(2|1)]$ case, although those results are somewhat less formally explained.

5.2.1 Matrix elements $\pi_\Lambda(K_4)$ for the $U_q[gl(3|1)]$ case

Firstly, we must normal order a list of 8 generator strings, cf. (46). We obtain:

$$K_4 \cdot \begin{pmatrix} \mathrm{Id} \\ E^4{}_3 \\ E^4{}_2 \\ E^4{}_1 \\ E^4{}_3 E^4{}_2 \\ E^4{}_3 E^4{}_1 \\ E^4{}_2 E^4{}_1 \\ E^4{}_3 E^4{}_2 E^4{}_1 \end{pmatrix} \overset{(15)}{=} \begin{pmatrix} \mathrm{Id} \\ \overline{q} E^4{}_3 \\ \overline{q} E^4{}_2 \\ \overline{q} E^4{}_1 \\ \overline{q}^2 E^4{}_3 E^4{}_2 \\ \overline{q}^2 E^4{}_3 E^4{}_1 \\ \overline{q}^2 E^4{}_2 E^4{}_1 \\ \overline{q}^3 E^4{}_3 E^4{}_2 E^4{}_1 \end{pmatrix} \cdot K_4. \tag{54}$$

The action $K_4 \cdot |1\rangle$ is known explicitly from (32), that is, we have $K_4 \cdot |1\rangle = \overline{q}^\alpha |1\rangle$. Thus, we have:

$$K_4 \cdot \begin{pmatrix} |1\rangle \\ |2\rangle \\ |3\rangle \\ |4\rangle \\ |5\rangle \\ |6\rangle \\ |7\rangle \\ |8\rangle \end{pmatrix} \overset{(54)}{=} \overline{q}^\alpha \begin{pmatrix} 1 \\ \overline{q}\beta_2 E^4{}_3 \\ \overline{q}\beta_3 E^4{}_2 \\ \overline{q}\beta_4 E^4{}_1 \\ \overline{q}^2\beta_5 E^4{}_3 E^4{}_2 \\ \overline{q}^2\beta_6 E^4{}_3 E^4{}_1 \\ \overline{q}^2\beta_7 E^4{}_2 E^4{}_1 \\ \overline{q}^3\beta_8 E^4{}_3 E^4{}_2 E^4{}_1 \end{pmatrix} \cdot |1\rangle \overset{(46)}{=} \overline{q}^\alpha \begin{pmatrix} |1\rangle \\ \overline{q}\,|2\rangle \\ \overline{q}\,|3\rangle \\ \overline{q}\,|4\rangle \\ \overline{q}^2\,|5\rangle \\ \overline{q}^2\,|6\rangle \\ \overline{q}^2\,|7\rangle \\ \overline{q}^3\,|8\rangle \end{pmatrix}.$$

Installing this information into (53), we discover:

$$\pi(K_4) = \overline{q}^\alpha \left(\begin{array}{l} |1\rangle\langle 1| + \overline{q}\,(|2\rangle\langle 2| + |3\rangle\langle 3| + |4\rangle\langle 4|) \\ + \overline{q}^2\,(|5\rangle\langle 5| + |6\rangle\langle 6| + |7\rangle\langle 7|) + \overline{q}^3\,|8\rangle\langle 8| \end{array} \right).$$

Note that this process *doesn't* actually require explicit knowledge of the β_i presented in (52). This is a feature of the evaluation of matrix elements of Cartan generators; the more usual situation appears in the next example (i.e. §5.2.2). Replacing $|i\rangle\langle j|$ with the elementary matrix $e^i_j \in M_{2^{mn}}$, we have:

$$\pi(K_4) = \overline{q}^\alpha \left(e^1_1 + \overline{q}\left(e^2_2 + e^3_3 + e^4_4\right) + \overline{q}^2\left(e^5_5 + e^6_6 + e^7_7\right) + \overline{q}^3 e^8_8 \right),$$

that is:

$$\pi(K_4) = \overline{q}^\alpha \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \overline{q} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \overline{q} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{q} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \overline{q}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \overline{q}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \overline{q}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \overline{q}^3 \end{pmatrix}.$$

5.2.2 Matrix elements $\pi_\Lambda(E^3{}_4)$ for the $U_q[gl(3|1)]$ case

This is a more interesting case than that of K_4. This time, the normal ordering of the list of 8 generator strings yields:

$$
E^3{}_4 \cdot
\begin{pmatrix}
\text{Id} \\
E^4{}_3 \\
E^4{}_2 \\
E^4{}_1 \\
E^4{}_3 E^4{}_2 \\
E^4{}_3 E^4{}_1 \\
E^4{}_2 E^4{}_1 \\
E^4{}_3 E^4{}_2 E^4{}_1
\end{pmatrix}
=
\begin{pmatrix}
E^3{}_4 \\
-E^4{}_3 E^3{}_4 + \overline{\Delta}(\overline{K}_4 K_3 - K_4 \overline{K}_3) \\
-E^4{}_2 E^3{}_4 + E^3{}_2 K_4 \overline{K}_3 \\
-E^4{}_1 E^3{}_4 + E^3{}_1 K_4 \overline{K}_3 \\
\overline{\Delta} E^4{}_2 (q\overline{K}_4 K_3 - \overline{q} K_4 \overline{K}_3) + E^4{}_3 (E^4{}_2 E^3{}_4 - E^3{}_2 K_4 \overline{K}_3) \\
\overline{\Delta} E^4{}_1 (q\overline{K}_4 K_3 - \overline{q} K_4 \overline{K}_3) + E^4{}_3 (E^4{}_1 E^3{}_4 - E^3{}_1 K_4 \overline{K}_3) \\
E^4{}_2 E^4{}_1 E^3{}_4 + \overline{q} E^4{}_1 E^3{}_2 K_4 \overline{K}_3 - E^4{}_2 E^3{}_1 K_4 \overline{K}_3 \\
-E^4{}_3 (E^4{}_2 E^4{}_1 E^3{}_4 + \overline{q} E^4{}_1 E^3{}_2 K_4 \overline{K}_3 - E^4{}_2 E^3{}_1 K_4 \overline{K}_3) \\
\quad + \overline{\Delta} E^4{}_2 E^4{}_1 (q^2 \overline{K}_4 K_3 - \overline{q}^2 K_4 \overline{K}_3)
\end{pmatrix},
$$

where we have again written $\Delta = q - \overline{q}$ and $\overline{\Delta} = \Delta^{-1}$ (cf. (10)), and done a little judicious factoring to improve readability. Next, we know that any terms which end with raising generators (e.g. $E^3{}_4$) will annihilate $|1\rangle$. Further, any term which contains an even lowering generator immediately to the left of a terminal string of Cartan generators[10] will annihilate $|1\rangle$, as the action of the Cartan generators is purely scalar. Omitting all such terms, we may thus write:

$$
E^3{}_4 \cdot
\begin{pmatrix}
\text{Id} \\
E^4{}_3 \\
E^4{}_2 \\
E^4{}_1 \\
E^4{}_3 E^4{}_2 \\
E^4{}_3 E^4{}_1 \\
E^4{}_2 E^4{}_1 \\
E^4{}_3 E^4{}_2 E^4{}_1
\end{pmatrix}
\cdot |1\rangle =
\begin{pmatrix}
0 \\
\overline{\Delta}(\overline{K}_4 K_3 - K_4 \overline{K}_3) \\
0 \\
0 \\
\overline{\Delta} E^4{}_2 (q\overline{K}_4 K_3 - \overline{q} K_4 \overline{K}_3) \\
\overline{\Delta} E^4{}_1 (q\overline{K}_4 K_3 - \overline{q} K_4 \overline{K}_3) \\
0 \\
\overline{\Delta} E^4{}_2 E^4{}_1 (q^2 \overline{K}_4 K_3 - \overline{q}^2 K_4 \overline{K}_3)
\end{pmatrix}
\cdot |1\rangle
$$

Again, the action $K_a^N \cdot |1\rangle$ is known explicitly from (32), that is, we have $K_3^\pm \cdot |1\rangle = |1\rangle$ and, as above, that $K_4^\pm \cdot |1\rangle = q^{\mp\alpha} |1\rangle$. Thus, we have:

$$
E^3{}_4 \cdot
\begin{pmatrix}
\text{Id} \\
E^4{}_3 \\
E^4{}_2 \\
E^4{}_1 \\
E^4{}_3 E^4{}_2 \\
E^4{}_3 E^4{}_1 \\
E^4{}_2 E^4{}_1 \\
E^4{}_3 E^4{}_2 E^4{}_1
\end{pmatrix}
\cdot |1\rangle =
\begin{pmatrix}
0 \\
[\alpha]_q \\
0 \\
0 \\
[\alpha + 1]_q E^4{}_2 \\
[\alpha + 1]_q E^4{}_1 \\
0 \\
[\alpha + 2]_q E^4{}_2 E^4{}_1
\end{pmatrix}
\cdot |1\rangle .
\tag{55}
$$

---

[10]For example, the last product contains $E^4{}_3 E^4{}_2 E^3{}_1 K_4 \overline{K}_3$, which contains the even lowering generator $E^3{}_1$ immediately to the left of a terminal string of Cartan generators.

This time explicitly installing information of the β_i from (52), we have:

$$
E^3{}_4 \cdot
\begin{pmatrix}
|1\rangle \\
|2\rangle \\
|3\rangle \\
|4\rangle \\
|5\rangle \\
|6\rangle \\
|7\rangle \\
|8\rangle
\end{pmatrix}
\overset{(46)}{=}
E^3{}_4 \cdot
\begin{pmatrix}
\beta_1 \\
\beta_2 E^4{}_3 \\
\beta_3 E^4{}_2 \\
\beta_4 E^4{}_1 \\
\beta_5 E^4{}_3 E^4{}_2 \\
\beta_6 E^4{}_3 E^4{}_1 \\
\beta_7 E^4{}_2 E^4{}_1 \\
\beta_8 E^4{}_3 E^4{}_2 E^4{}_1
\end{pmatrix}
\cdot |1\rangle
\overset{(55)}{=}
\begin{pmatrix}
0 \\
\beta_2 [\alpha]_q |1\rangle \\
0 \\
0 \\
\beta_5 [\alpha+1]_q E^4{}_2 \cdot |1\rangle \\
\beta_6 [\alpha+1]_q E^4{}_1 \cdot |1\rangle \\
0 \\
\beta_8 [\alpha+2]_q E^4{}_2 E^4{}_1 \cdot |1\rangle
\end{pmatrix}
$$

$$
\overset{(46)}{=}
\begin{pmatrix}
0 \\
\beta_2 [\alpha]_q |1\rangle \\
0 \\
0 \\
\beta_5 \overline{\beta}_3 [\alpha+1]_q |3\rangle \\
\beta_6 \overline{\beta}_4 [\alpha+1]_q |4\rangle \\
0 \\
\beta_8 \overline{\beta}_7 [\alpha+2]_q |7\rangle
\end{pmatrix}
\overset{(52)}{=}
\begin{pmatrix}
0 \\
[\alpha]_q^{\frac{1}{2}} |1\rangle \\
0 \\
0 \\
[\alpha+1]_q^{\frac{1}{2}} |3\rangle \\
[\alpha+1]_q^{\frac{1}{2}} |4\rangle \\
0 \\
[\alpha+2]_q^{\frac{1}{2}} |7\rangle
\end{pmatrix}. \tag{56}
$$

Note that there is a subtle point in the second application of (46), in that we are using it *implicitly*: this point wasn't so clear when we computed $\pi(K_4)$. Whilst this is simple enough for a human to perform, computer programs require *explicit* algorithms. In practice, what this means is that we must invert (46) to provide a list of transformation rules for strings of odd lowering generators acting on $|1\rangle$ that allow us to recover the $|i\rangle$. These rules must be carefully coded, and applied in *reverse* order to (46), to ensure that we collect the longest strings first. Again, the ordering of the $|i\rangle$ facilitates this. To illustrate, we want to apply, *in order*, the following rules:

$$
\left\{
\begin{array}{rcl}
E^4{}_3 E^4{}_2 E^4{}_1 \cdot |1\rangle & \mapsto & \overline{\beta}_8 |8\rangle \\
E^4{}_2 E^4{}_1 \cdot |1\rangle & \mapsto & \overline{\beta}_7 |7\rangle \\
E^4{}_3 E^4{}_1 \cdot |1\rangle & \mapsto & \overline{\beta}_6 |6\rangle \\
E^4{}_3 E^4{}_2 \cdot |1\rangle & \mapsto & \overline{\beta}_5 |5\rangle \\
E^4{}_1 \cdot |1\rangle & \mapsto & \overline{\beta}_4 |4\rangle \\
E^4{}_2 \cdot |1\rangle & \mapsto & \overline{\beta}_3 |3\rangle \\
E^4{}_3 \cdot |1\rangle & \mapsto & \overline{\beta}_2 |2\rangle
\end{array}
\right\}.
$$

Installing the information in (56) into (53), we discover, again substituting e^i_j for $|i\rangle\langle j|$:

$$
\begin{aligned}
\pi(E^3{}_4) &= [\alpha]_q^{\frac{1}{2}} |1\rangle\langle 2| + [\alpha+1]_q^{\frac{1}{2}} |3\rangle\langle 5| + [\alpha+1]_q^{\frac{1}{2}} |4\rangle\langle 6| + [\alpha+2]_q^{\frac{1}{2}} |7\rangle\langle 8| \\
&= [\alpha]_q^{\frac{1}{2}} e^1_2 + [\alpha+1]_q^{\frac{1}{2}} e^3_5 + [\alpha+1]_q^{\frac{1}{2}} e^4_6 + [\alpha+2]_q^{\frac{1}{2}} e^7_8,
\end{aligned}
$$

and interpreting e^i_j as an elementary matrix:

$$\pi(E^3{}_4) = \begin{pmatrix} 0 & [\alpha]_q^{\frac{1}{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & [\alpha+1]_q^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & [\alpha+1]_q^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & [\alpha+2]_q^{\frac{1}{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

5.2.3 Matrix elements for the remaining generators

Having found the matrix elements for the *simple raising* generators $\pi(E^a{}_b)$, we may immediately write down the matrix elements for the corresponding *simple lowering* generators $\pi(E^b{}_a)$, as these are necessarily the transposes of $\pi(E^a{}_b)$.

To illustrate, for the $U_q[gl(3|1)]$ case, we have:

$$\pi(E^4{}_3) = [\alpha]_q^{\frac{1}{2}} e_1^2 + [\alpha+1]_q^{\frac{1}{2}} e_3^5 + [\alpha+1]_q^{\frac{1}{2}} e_4^6 + [\alpha+2]_q^{\frac{1}{2}} e_7^8.$$

Beyond this, we may construct matrix elements for the *nonsimple* generators from those for the simple ones by (recursively) applying π to (8a), viz:

$$\pi(E^a{}_b) = \pi(E^a{}_c)\pi(E^c{}_b) - q_c^{S^a_b}\pi(E^c{}_b)\pi(E^a{}_c),$$

although this may not yield particularly *useful* results. Note that we can in fact write down the matrix elements for any (simple and nonsimple) *lowering* generators directly from their corresponding raising generators, as we have, for $a > b$: $\pi(E^a{}_b) = \pi(E^b{}_a)^{T_q}$, where the 'q transpose' T_q indicates the combination of the transpose and the mapping $q \mapsto \overline{q}$. This follows from inspection of the way that (8a) depends on q; indeed it is trivial for simple generators. Illustrations are visible below (and in Appendix B.1); the reader should note that the q bracket is invariant under $q \mapsto \overline{q}$.

For completeness, we list the matrix elements of the $U_q[gl(3|1)]$ *simple* generators (the matrix elements of *all* the generators are listed in Appendix B.1):

$$\left.\begin{array}{ll} \text{(a)} & \pi(K_1) = e_1^1 + e_2^2 + e_3^3 + \overline{q}e_4^4 + e_5^5 + \overline{q}e_6^6 + \overline{q}e_7^7 + \overline{q}e_8^8 \\ \text{(b)} & \pi(K_2) = e_1^1 + e_2^2 + \overline{q}e_3^3 + e_4^4 + \overline{q}e_5^5 + e_6^6 + \overline{q}e_7^7 + \overline{q}e_8^8 \\ \text{(c)} & \pi(K_3) = e_1^1 + \overline{q}e_2^2 + e_3^3 + e_4^4 + \overline{q}e_5^5 + \overline{q}e_6^6 + e_7^7 + \overline{q}e_8^8 \\ \text{(d)} & \pi(K_4) = \overline{q}^{\alpha}e_1^1 + \overline{q}^{\alpha+1}(e_2^2 + e_3^3 + e_4^4) + \overline{q}^{\alpha+2}(e_5^5 + e_6^6 + e_7^7) + \overline{q}^{\alpha+3}e_8^8 \\ \text{(e)} & \pi(E^1{}_2) = -e_4^3 - e_6^5 \\ \text{(f)} & \pi(E^2{}_1) = -e_3^4 - e_5^6 \\ \text{(g)} & \pi(E^2{}_3) = -e_3^2 - e_7^6 \\ \text{(h)} & \pi(E^3{}_2) = -e_2^3 - e_6^7 \\ \text{(i)} & \pi(E^3{}_4) = [\alpha]_q^{\frac{1}{2}} e_2^1 + [\alpha+1]_q^{\frac{1}{2}} e_5^3 + [\alpha+1]_q^{\frac{1}{2}} e_6^4 + [\alpha+2]_q^{\frac{1}{2}} e_8^7 \\ \text{(j)} & \pi(E^4{}_3) = [\alpha]_q^{\frac{1}{2}} e_1^2 + [\alpha+1]_q^{\frac{1}{2}} e_3^5 + [\alpha+1]_q^{\frac{1}{2}} e_4^6 + [\alpha+2]_q^{\frac{1}{2}} e_7^8. \end{array}\right\} \quad (57)$$

6 The submodules $V_k \subset V \otimes V$

We now turn our attention to the tensor product module $V \otimes V$. Where V has a basis $B = \{|i\rangle\}_{i=1}^{2mn}$, the 2^{2mn} dimensional $V \otimes V$ has a natural basis $\{|i\rangle \otimes |j\rangle\}_{i,j=1}^{2mn}$, which inherits a weight system and a grading from V:

$$\begin{aligned}
\mathrm{wt}(|i\rangle \otimes |j\rangle) &\triangleq \mathrm{wt}(|i\rangle) + \mathrm{wt}(|j\rangle) && (58) \\
[|i\rangle \otimes |j\rangle] &\triangleq [|i\rangle] + [|j\rangle] \pmod 2.
\end{aligned}$$

To build an R matrix acting on $V \otimes V$, we will use an alternative, orthonormal weight basis $\mathfrak{B}$ for $V \otimes V$, which corresponds to the (known) decomposition of $V \otimes V$ into irreducible $U_q[gl(m|n)]$ submodules. The basis vectors of $\mathfrak{B}$ are expressed as linear combinations $\theta_{ij}(|i\rangle \otimes |j\rangle)$, where the coefficients θ_{ij} are in general algebraic expressions in q and α. Before proceeding, we introduce some machinery for dealing with the tensor products.

6.1 Tensor product representation tools

We first introduce the action of $U_q[gl(m|n)] \otimes U_q[gl(m|n)]$ on $V \otimes V$. Where Y is an homogeneous $U_q[gl(m|n)]$ element, we define:

$$(X \otimes Y) \cdot (|i\rangle \otimes |j\rangle) \triangleq (-)^{[Y][i]}(X \cdot |i\rangle \otimes Y \cdot |j\rangle), \tag{59}$$

and extend by linearity to all of $U_q[gl(m|n)] \otimes U_q[gl(m|n)]$; note that we have written $[i] \equiv [|i\rangle]$ for readability. This action is compatible with the grading.

Next, we define a basis $\{\langle i| \otimes \langle j|\}_{i,j=1}^{2mn}$ for $(V \otimes V)^*$ (the dual to $V \otimes V$) by dualising each of the elements of the basis $\{|i\rangle \otimes |j\rangle\}_{i,j=1}^{2mn}$:

$$\langle i| \otimes \langle j| \triangleq (-)^{[i][j]}(|i\rangle \otimes |j\rangle)^\dagger, \tag{60}$$

where we intend $(|i\rangle \otimes |j\rangle)^\dagger = |i\rangle^\dagger \otimes |j\rangle^\dagger$. The conjugate is extended by linearity:

$$(A\,|i\rangle \otimes |j\rangle + B\,|k\rangle \otimes |l\rangle)^\dagger \triangleq A^*(|i\rangle \otimes |j\rangle)^\dagger + B^*(|k\rangle \otimes |l\rangle)^\dagger,$$

for scalars A and B, where A^* is the complex conjugate of A.

The multiplication operations between the dual bases are given by:

$$\begin{aligned}
(|i\rangle \otimes |j\rangle) \cdot (\langle k| \otimes \langle l|) &\triangleq (-)^{[j][k]}(|i\rangle\langle k| \otimes |j\rangle\langle l|) && (61) \\
(\langle i| \otimes \langle j|) \cdot (|k\rangle \otimes |l\rangle) &\triangleq (-)^{[j][k]}\delta_k^i \delta_l^j, && (62)
\end{aligned}$$

analagously to (59).

Thus, we have the natural inner product on the basis $\{|i\rangle \otimes |j\rangle\}_{i,j=1}^{2mn}$:

$$(|i\rangle \otimes |j\rangle, |k\rangle \otimes |l\rangle) \triangleq (|i\rangle \otimes |j\rangle)^\dagger \cdot (|k\rangle \otimes |l\rangle), \tag{63}$$

which behaves as expected:

$$\begin{aligned}
(|i\rangle \otimes |j\rangle, |k\rangle \otimes |l\rangle) &\overset{(60)}{=} (-)^{[i][j]}(\langle i| \otimes \langle j|) \cdot (|k\rangle \otimes |l\rangle) \\
&= (-)^{[i][j]}(-)^{[j][k]}(\langle i|k\rangle \otimes \langle j|l\rangle) \\
&= (-)^{[j]([i]+[k])}\delta_k^i \delta_l^j = \delta_k^i \delta_l^j.
\end{aligned}$$

Lastly, we will often use the shorthand $|i \otimes j\rangle \triangleq |i\rangle \otimes |j\rangle$ and $\langle i \otimes j| \triangleq \langle i| \otimes \langle j|$.

6.2 The orthogonal decomposition of $V \otimes V$

For our modules $V \equiv V_\Lambda$, the orthogonal decomposition of $V \otimes V$ is known, and contains no multiplicities [11, 15]. To describe it, we introduce a little notation. What follows is strictly true only for $m \geqslant n$, but the natural isomorphism between $U_q[gl(m|n)]$ and $U_q[gl(n|m)]$ shows that we need not consider the other case $n \geqslant m$.

For any $0 \leqslant N \leqslant mn$, let γ be a nonincreasing sequence of nonnegative integers[11] $\gamma_1, \gamma_2, \ldots, \gamma_r$, where the γ_k satisfy $\sum_{k=1}^{r} \gamma_k = N$. We then define a *Young diagram* $[\gamma] = [\gamma_1, \gamma_2, \ldots, \gamma_r]$, to be *allowable* if it has at most n columns and m rows, viz $r \leqslant m$ and, for each k, $\gamma_k \leqslant n$. To each such allowable diagram, we associate a $gl(m|n)$ weight λ_γ:

$$\lambda_\gamma = (\dot{0}_{m-r}, -\gamma_r, \ldots, -\gamma_2, -\gamma_1 \mid \dot{r}_{\gamma_r}, (r \overset{.}{-} 1)_{\gamma_{r-1}-\gamma_r}, \ldots, \dot{1}_{\gamma_1-\gamma_2}, \dot{0}_{n-\gamma_1}). \quad (64)$$

Then, for our specific representations with $\Lambda = (\dot{0}_m|\dot{\alpha}_n)$, modulo the comments on limiting α in §5 (viz α must be real and either $\alpha > 1 - n$ or $\alpha < 1 - m$), we have the following irreducible decomposition of V_Λ:

$$V_\Lambda = \bigoplus_{[\gamma]} V_{\Lambda+\lambda_\gamma}^0,$$

where the sum is over all possible allowable Young diagrams, and $V_{\Lambda+\lambda_\gamma}^0$ is a $U_q[gl(m) \oplus gl(n)]$ module of highest weight $\Lambda + \lambda_\gamma$ and $\mathbb{Z}$ graded level N. More interestingly, we also have the following decomposition:

$$V_\Lambda \otimes V_\Lambda = \bigoplus_{[\gamma]} V_{2\Lambda+\lambda_\gamma}, \quad (65)$$

where again the sum is over all possible allowable Young diagrams, and $V_{2\Lambda+\lambda_\gamma}$ is a $U_q[gl(m|n)]$ module of highest weight $2\Lambda + \lambda_\gamma$ and $\mathbb{Z}$ graded level N.

And now, we introduce an abuse of notation. Instead of the explicit (65), we shall often write:

$$V \otimes V = \bigoplus_{k} V_k, \quad (66)$$

where the submodule V_k has highest weight λ_k, and the generic index k runs over some appropriate index set. In the *special* case $n = 1$, the decomposition of (65) becomes:

$$V \otimes V = \bigoplus_{k=0}^{m} V_k, \quad (67)$$

where V_k has highest weight $\lambda_k = (\dot{0}_{m-k}, (\overset{.}{-}1)_k \mid 2\alpha + k)$, thus the submodules $V_0, V_1, \ldots, V_m$ are ordered by increasing $\mathbb{Z}$ graded level (which, in this case, is in fact, k). We shall be using (67) to illustrate specific examples later.

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[11] We apologise for overloading γ – this one is a sequence, not a root.

In the following subsections we describe the construction of orthonormal weight bases $\mathfrak{B}_k = \{\left|\Psi_l^k\right\rangle\}_{l=1}^{\dim(V_k)}$ for each V_k, where $\mathrm{span}(\bigcup_k \mathfrak{B}_k) = V \otimes V$. Fixing k, each $\left|\Psi_l^k\right\rangle$ is a linear combination of terms of the form $\theta_{ij}\left|i\otimes j\right\rangle$. As we may safely define the duals $\left\langle\Psi_l^k\right| \triangleq \left|\Psi_l^k\right\rangle^\dagger$ (as expansion of the conjugate is possible using (60)), it is thus meaningful to orthonormalise the $\left|\Psi_l^k\right\rangle$.

Firstly, in §6.3, we determine a highest weight vector $\left|\Psi_1^k\right\rangle$ for V_k; which we use as the starting point for the KIMC. Next, in §6.4 we build a basis $\mathfrak{B}_k^0$ for the even subalgebra submodule V_k^0. In this case, as distinct from that of V^0 in §5, V_0^k may not be one-dimensional (although it contains no weight multiplicities), so this construction is nontrivial, although it turns out to be quite straightforward. Lastly, in §6.5 and §6.6, we construct $\mathfrak{B}_k$ by the actions of the 2^{mn} possible combinations of ordered nonrepeated lowering generators on $\mathfrak{B}_k^0$. A subtlety in this case is that V_k in general contains weight multiplicities, so that we must employ a Gram–Schmidt process to orthonormalise it.

6.3 A highest weight vector $\left|\Psi_1^k\right\rangle$ for V_k

We begin the construction of $\mathfrak{B}_k$ with the deduction of a highest weight vector $\left|\Psi_1^k\right\rangle$, of weight λ_k. As we know the weights of the $\left|i\right\rangle$, using (58), we may immediately write down:

$$\left|\Psi_1^k\right\rangle = \sum_{ij}\theta_{ij}\left|i\otimes j\right\rangle, \tag{68}$$

where the sum is only over i,j such that $\mathrm{wt}(\left|\Psi_1^k\right\rangle) = \lambda_k = \mathrm{wt}(\left|i\right\rangle) + \mathrm{wt}(\left|j\right\rangle)$ is satisfied (i.e. we don't know in advance how many terms there are in the sum). The coefficients θ_{ij} are scalar expressions in q and α, which we shall determine by the following.

1. We demand that $\left|\Psi_1^k\right\rangle$ be annihilated by the actions of (the coproducts of) all raising generators. (Actually, it is sufficient that it be annihilated by all *simple* raising generators.) As $\dim(V_k^0) \neq 1$ in general, $\left|\Psi_1^k\right\rangle$ may not be annihilated by the actions of *even* lowering generators.

2. We demand that $\left|\Psi_1^k\right\rangle$ be normalised. Examination of (68) shows that, using (63), the magnitude of $\left|\Psi_1^k\right\rangle$ is $\left(\sum_{ij}(-)^{[i][j]}\theta_{ij}^*\theta_{ij}\right)^{\frac{1}{2}}$, where the sum is again only over appropriate i,j. As the θ_{ij} are only determined up to arbitrary phase factors, we may replace $\theta_{ij}^*\theta_{ij}$ with θ_{ij}^2, hence we demand:

$$\sum_{ij}(-)^{[i][j]}\theta_{ij}^2 = 1. \tag{69}$$

It turns out that these considerations always yield exactly enough constraints to uniquely determine $\left|\Psi_1^k\right\rangle$ (well, up to an unimportant phase factor).

Before proceeding, we observe that $\left|\Psi_1^0\right\rangle$, the highest weight vector of the first module V_0, is necessarily $\left|1\otimes 1\right\rangle$. This provides a check on the methods used to determine the other $\left|\Psi_1^k\right\rangle$. More generally, we must set up a linear system to determine the coefficients θ_{ij}, and again, MATHEMATICA is well-suited to this.

6.3.1 Illustration: $\left|\Psi_1^2\right\rangle$ for $U_q[gl(3|1)]$

We now use $\left|\Psi_1^2\right\rangle$ for $U_q[gl(3|1)]$ to illustrate the entire process. Using the notation of (67), we have 4 submodules V_k; their highest weights, dimensions (obtained from (34)) and suitable $\left|\Psi_1^k\right\rangle$ are presented in Table 3. Note that for $n = 1$, for $\Lambda = (\dot{0}_m|\alpha)$, in fact (34) degenerates to $\dim(V_k^0) = \binom{m}{k}$.

k	λ_k	dim V_k^0	V_k	$\left|\Psi_1^k\right\rangle$
0	$(\ 0,\ \ 0,\ \ 0\ \|\ 2\alpha\ \ \ )$	1	8	$\theta_{11}\left|1{\otimes}1\right\rangle$
1	$(\ 0,\ \ 0, -1\ \|\ 2\alpha+1)$	3	24	$\theta_{12}\left|1{\otimes}2\right\rangle + \theta_{21}\left|2{\otimes}1\right\rangle$
2	$(\ 0, -1, -1\ \|\ 2\alpha+2)$	3	24	$\theta_{15}\left|1{\otimes}5\right\rangle + \theta_{51}\left|5{\otimes}1\right\rangle$ $+\theta_{23}\left|2{\otimes}3\right\rangle + \theta_{32}\left|3{\otimes}2\right\rangle$
3	$(-1, -1, -1\ \|\ 2\alpha+3)$	1	8	$\theta_{18}\left|1{\otimes}8\right\rangle + \theta_{81}\left|8{\otimes}1\right\rangle$ $+\theta_{27}\left|2{\otimes}7\right\rangle + \theta_{72}\left|7{\otimes}2\right\rangle$ $+\theta_{36}\left|3{\otimes}6\right\rangle + \theta_{63}\left|6{\otimes}3\right\rangle$ $+\theta_{45}\left|4{\otimes}5\right\rangle + \theta_{54}\left|5{\otimes}4\right\rangle$

Table 3: The tensor product submodules for the $U_q[gl(3|1)]$ case.

From Table 3, for $\left|\Psi_1^2\right\rangle$, we must determine 4 coefficients:

$$\left|\Psi_1^2\right\rangle = \theta_{15}\left|1{\otimes}5\right\rangle + \theta_{51}\left|5{\otimes}1\right\rangle + \theta_{23}\left|2{\otimes}3\right\rangle + \theta_{32}\left|3{\otimes}2\right\rangle. \tag{70}$$

The $U_q[gl(3|1)]$ simple raising generator set is $SRGS = \{E^1{}_2, E^2{}_3, E^3{}_4\}$, and from (25b), the coproducts are:

$$\left.\begin{aligned}
\Delta(E^1{}_2) &= (E^1{}_2 \otimes \overline{K_1^{\frac{1}{2}}} K_2^{\frac{1}{2}}) + (K_1^{\frac{1}{2}} \overline{K_2^{\frac{1}{2}}} \otimes E^1{}_2) \\
\Delta(E^2{}_3) &= (E^2{}_3 \otimes \overline{K_2^{\frac{1}{2}}} K_3^{\frac{1}{2}}) + (K_2^{\frac{1}{2}} \overline{K_3^{\frac{1}{2}}} \otimes E^2{}_3) \\
\Delta(E^3{}_4) &= (E^3{}_4 \otimes \overline{K_3^{\frac{1}{2}}} K_4^{\frac{1}{2}}) + (K_3^{\frac{1}{2}} \overline{K_4^{\frac{1}{2}}} \otimes E^3{}_4),
\end{aligned}\right\} \tag{71}$$

so the actions that we want are:

$$\begin{pmatrix} \Delta(E^1{}_2) \\ \Delta(E^2{}_3) \\ \Delta(E^3{}_4) \end{pmatrix} \cdot \left|\Psi_1^2\right\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

To evaluate these products, we take the known matrix elements of the underlying representation (57), and substitute these into the evaluations of the coproducts (71). Thus, for the example $\Delta(E^3{}_4) \cdot \left|\Psi_1^2\right\rangle$, we find:

$$\Delta(E^3{}_4) \cdot \left|\Psi_1^2\right\rangle = \left((E^3{}_4 \otimes \overline{K_3^{\frac{1}{2}}} K_4^{\frac{1}{2}}) + (K_3^{\frac{1}{2}} \overline{K_4^{\frac{1}{2}}} \otimes E^3{}_4)\right) \cdot$$
$$(\theta_{15}\left|1\right\rangle \otimes \left|5\right\rangle + \theta_{51}\left|5\right\rangle \otimes \left|1\right\rangle + \theta_{23}\left|2\right\rangle \otimes \left|3\right\rangle + \theta_{32}\left|3\right\rangle \otimes \left|2\right\rangle).$$

To illustrate the multiplication:

$$\left(E^3{}_4 \otimes \overline{K}_3^{\frac{1}{2}} K_4^{\frac{1}{2}}\right) \cdot \left(|5\rangle \otimes |1\rangle\right) \overset{(59)}{=} E^3{}_4 \cdot |5\rangle \otimes \overline{K}_3^{\frac{1}{2}} K_4^{\frac{1}{2}} \cdot |1\rangle \overset{(57i)}{=} \overline{q}^{\frac{\alpha}{2}} A_1^{\frac{1}{2}} |3\rangle \otimes |1\rangle .$$

At this point, to save space, we introduce a little more notation, which eliminates the q brackets altogether:

$$A_{i,j}^z \triangleq ([i\alpha + j]_q)^z, \qquad \text{where } z \in \{\tfrac{1}{2}, 1\} \tag{72}$$

$$C_{i,j} \triangleq q^{i\alpha+j} + \overline{q}^{i\alpha+j}, \tag{73}$$

where $i, j \geqslant 0$. In these expressions, if $i = 1$ we shall simply omit it, that is, we intend: $C_j \triangleq C_{1,j} = q^{\alpha+j} + \overline{q}^{\alpha+j}$ and $A_j^z \triangleq A_{1,j}^z = ([\alpha + j]_q)^z$. Occasionally, we will write $\overline{A}_{i,j}^z = A_{i,j}^{-z}$ and $\overline{C}_{i,j} = (C_{i,j})^{-1}$.

Using this notation, we have:

$$\Delta(E^3{}_4) \cdot \left|\Psi_1^2\right\rangle = \left(\left(E^3{}_4 \otimes \overline{K}_3^{\frac{1}{2}} K_4^{\frac{1}{2}}\right) + \left(K_3^{\frac{1}{2}} \overline{K}_4^{\frac{1}{2}} \otimes E^3{}_4\right)\right) \cdot$$

$$\left(\theta_{15} |1\rangle \otimes |5\rangle + \theta_{51} |5\rangle \otimes |1\rangle + \theta_{23} |2\rangle \otimes |3\rangle + \theta_{32} |3\rangle \otimes |2\rangle\right)$$

$$\overset{(57)}{=} \left(\theta_{15} q^{\frac{\alpha}{2}} A_1^{\frac{1}{2}} + \theta_{23} \overline{q}^{\frac{\alpha}{2}+\frac{1}{2}} A_0^{\frac{1}{2}}\right) \left(|1\rangle \otimes |3\rangle\right)$$

$$+ \left(\theta_{51} \overline{q}^{\frac{\alpha}{2}} A_1^{\frac{1}{2}} - \theta_{32} q^{\frac{\alpha}{2}+\frac{1}{2}} A_0^{\frac{1}{2}}\right) \left(|3\rangle \otimes |1\rangle\right),$$

and, altogether, for the three $SRGS$ generators, we have:

$$\begin{pmatrix} \Delta(E^1{}_2) \\ \Delta(E^2{}_3) \\ \Delta(E^3{}_4) \end{pmatrix} \cdot \left|\Psi_1^2\right\rangle = \begin{pmatrix} 0 \\ \left(-\theta_{23} q^{\frac{1}{2}} - \theta_{32} \overline{q}^{\frac{1}{2}}\right) \left(|2\rangle \otimes |2\rangle\right) \\ \begin{pmatrix} \left(\theta_{15} q^{\frac{\alpha}{2}} A_1^{\frac{1}{2}} + \theta_{23} \overline{q}^{\frac{\alpha}{2}+\frac{1}{2}} A_0^{\frac{1}{2}}\right) \left(|1\rangle \otimes |3\rangle\right) \\ + \left(\theta_{51} \overline{q}^{\frac{\alpha}{2}} A_1^{\frac{1}{2}} - \theta_{32} q^{\frac{\alpha}{2}+\frac{1}{2}} A_0^{\frac{1}{2}}\right) \left(|3\rangle \otimes |1\rangle\right) \end{pmatrix} \end{pmatrix} .$$

As each component of the RHS must be zero, and the $|i\rangle \otimes |j\rangle$ are linearly independent, we thus obtain a net 3 *linear* constraints on the θ_{ij} from this set:

$$-q^{\frac{1}{2}} \theta_{23} - \overline{q}^{\frac{1}{2}} \theta_{32} = 0$$

$$q^{\frac{\alpha}{2}} A_1^{\frac{1}{2}} \theta_{15} + \overline{q}^{\frac{\alpha}{2}+\frac{1}{2}} A_0^{\frac{1}{2}} \theta_{23} = 0$$

$$\overline{q}^{\frac{\alpha}{2}} A_1^{\frac{1}{2}} \theta_{51} - q^{\frac{\alpha}{2}+\frac{1}{2}} A_0^{\frac{1}{2}} \theta_{32} = 0,$$

better written in matrix form as:

$$\begin{pmatrix} 0 & 0 & -q^{\frac{1}{2}} & -\overline{q}^{\frac{1}{2}} \\ q^{\frac{\alpha}{2}} A_1^{\frac{1}{2}} & 0 & \overline{q}^{\frac{\alpha}{2}+\frac{1}{2}} A_0^{\frac{1}{2}} & 0 \\ 0 & \overline{q}^{\frac{\alpha}{2}} A_1^{\frac{1}{2}} & 0 & -q^{\frac{\alpha}{2}+\frac{1}{2}} A_0^{\frac{1}{2}} \end{pmatrix} \cdot \begin{pmatrix} \theta_{15} \\ \theta_{51} \\ \theta_{23} \\ \theta_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} . \tag{74}$$

Note that we have left the system exactly as supplied by the raw action of the raising generators. A human calculator might delete superfluous signs, and perhaps do some factorisation, but our MATHEMATICA code would require explicit instructions for this finicky and unnecessary work, so we omit it.

Thus, application of the requirement that $\left|\Psi_1^2\right\rangle$ be annihilated by the simple raising generators yields a linear system of 3 equations in 4 variables. A final constraint to completely determine the variables is now obtained by requiring that $\left|\Psi_1^2\right\rangle$ be normalised.

In our example, using (69) and (70), and recalling the gradings of the basis vectors (in Table 2), we thus have the *nonlinear* constraint:

$$\theta_{15}^2 + \theta_{51}^2 - \theta_{23}^2 - \theta_{32}^2 = 1. \tag{75}$$

Combining the information in (74) and (75), we are able to solve for the unknowns θ_{ij}, uniquely up to the usual overall phase factor. In practice, we may actually avoid the use of (75), by first feeding (74) to the MATHEMATICA equation solver, which returns us an answer with a free parameter (the first unknown θ_{ij}).[12] Setting that free parameter to unity, we obtain a suitable unnormalised $\left|\Psi_1^2\right\rangle$, which we may immediately normalise. To illustrate, we find:

$$\left|\Psi_1^2\right\rangle = \frac{\overline{q}^{\alpha+1}A_0^{\frac{1}{2}}}{C_1^{\frac{1}{2}}A_{2,1}^{\frac{1}{2}}} \left|1\otimes5\right\rangle + \frac{q^{\alpha+1}A_0^{\frac{1}{2}}}{C_1^{\frac{1}{2}}A_{2,1}^{\frac{1}{2}}} \left|5\otimes1\right\rangle - \frac{\overline{q}^{\frac{1}{2}}A_1^{\frac{1}{2}}}{C_1^{\frac{1}{2}}A_{2,1}^{\frac{1}{2}}} \left|2\otimes3\right\rangle + \frac{q^{\frac{1}{2}}A_1^{\frac{1}{2}}}{C_1^{\frac{1}{2}}A_{2,1}^{\frac{1}{2}}} \left|3\otimes2\right\rangle .$$

To complete the results, we have obtained the highest weight vectors for each of the 4 submodules for the $U_q[gl(3|1)]$ case. These are, after a little factoring:

$\left|\Psi_1^0\right\rangle = \left|1\otimes1\right\rangle$

$\left|\Psi_1^1\right\rangle = \overline{C}_0^{\frac{1}{2}}\left(q^{\frac{\alpha}{2}}\left|2\otimes1\right\rangle - \overline{q}^{\frac{\alpha}{2}}\left|1\otimes2\right\rangle\right)$

$\left|\Psi_1^2\right\rangle = \overline{C}_1^{\frac{1}{2}}\overline{A}_{2,1}^{\frac{1}{2}}\left(\overline{q}^{\alpha+1}A_0^{\frac{1}{2}}\left|1\otimes5\right\rangle + q^{\alpha+1}A_0^{\frac{1}{2}}\left|5\otimes1\right\rangle - \overline{q}^{\frac{1}{2}}A_1^{\frac{1}{2}}\left|2\otimes3\right\rangle + q^{\frac{1}{2}}A_1^{\frac{1}{2}}\left|3\otimes2\right\rangle\right)$

$\left|\Psi_1^3\right\rangle = \overline{C}_1^{\frac{1}{2}}\overline{C}_2^{\frac{1}{2}}\overline{A}_{2,3}^{\frac{1}{2}}\times$
$\quad\left(\begin{array}{l}-\overline{q}^{\frac{3\alpha}{2}+3}A_0^{\frac{1}{2}}\left|1\otimes8\right\rangle + q^{\frac{3\alpha}{2}+3}A_0^{\frac{1}{2}}\left|8\otimes1\right\rangle + \overline{q}^{\frac{\alpha}{2}+2}A_2^{\frac{1}{2}}\left|2\otimes7\right\rangle - q^{\frac{\alpha}{2}+2}A_2^{\frac{1}{2}}\left|7\otimes2\right\rangle \\ -\overline{q}^{\frac{\alpha}{2}+1}A_2^{\frac{1}{2}}\left|3\otimes6\right\rangle + q^{\frac{\alpha}{2}+1}A_2^{\frac{1}{2}}\left|6\otimes3\right\rangle + \overline{q}^{\frac{\alpha}{2}}A_2^{\frac{1}{2}}\left|4\otimes5\right\rangle - q^{\frac{\alpha}{2}}A_2^{\frac{1}{2}}\left|5\otimes4\right\rangle\end{array}\right).$

Observe the presence of *q graded symmetric combinations* of $\left|i\otimes j\right\rangle$ and $\left|j\otimes i\right\rangle$ in these expressions, viz patterns of the form $q^x\left|i\otimes j\right\rangle \pm \overline{q}^x\left|j\otimes i\right\rangle$. This feature appears repeatedly throughout the bases for the V_k.

So, at this stage, we have described how to construct normalised highest weight vectors $\left|\Psi_1^k\right\rangle$ for each V_k. An interesting outstanding point about our process is that the two demands that $\left|\Psi_1^k\right\rangle$ be annihilated by the $m+n-1$ simple raising generators and that it be normalised, yield *exactly* enough constraints to determine it uniquely (up to a phase factor). The reason for this balance lies buried in the combinatorics of the ways the weights of the underlying module V can be added to yield the weight λ_k.

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[12] In fact, it returns us *two* answers, differing by a (phase) factor of -1. We judiciously choose to ignore the second one.

6.4   A basis $\mathfrak{B}^0_k$ for V^0_k

Having determined $\left|\Psi^k_1\right\rangle$, we now apply the first stage of the KIMC to construct the basis $\mathfrak{B}^0_k$. That is, we construct basis vectors $\left|\Psi^k_j\right\rangle$ by the repeated action of the $m+n-2$ even simple lowering generators (a set which we call $ESLGS$)[13] on the known $\left|\Psi^k_1\right\rangle$. In our case, as V^0_k contains no weight multiplicities, vectors of distinct weights created in this way will naturally be orthogonal. At each stage, we must also check to see if newly minted vectors are scalar multiples of previously found ones. To facilitate this, we will normalise each vector as we create it, and we will also maintain our list of vectors in order of decreasing weight. In fact, (34) tells us $\dim(V^0_k)$, but we will build $\mathfrak{B}^0_k$ as if we didn't know this.

Recall that in §5.1, we created $\mathbb{Z}$ graded levels of B by repeated applications of the set of odd lowering generators to the set $\{|1\rangle\}$. Here, each application of $ESLGS$ generates an *ungraded* level, which we shall call L_i, where $L_0 = \{\left|\Psi^k_1\right\rangle\}$. We describe the process in the following algorithm:

$$i := 0$$
$$L_0 := \{\left|\Psi^k_1\right\rangle\}$$
$$\text{while } L_i \neq \emptyset$$
$$\qquad L_{i+1} := (\Delta(ESLGS) \cdot L_i) \setminus \{0\}$$
$$\qquad \text{normalise } L_{i+1}$$
$$\qquad L_{i+1} := L_{i+1} \setminus L$$
$$\qquad L := L \cup L_{i+1}$$
$$\qquad \text{sort } L \text{ by decreasing weight}$$
$$\qquad \text{increment } i$$
$$\mathfrak{B}^0_k := L$$

Note that we have taken notational liberties by writing $\Delta(ESLGS) \cdot L_i$. We do this as it is natural to apply functions to lists in MATHEMATICA.

Of particular interest here is that the evaluation of the algebra-module action in the tensor product case utilises the information encoded in the matrix elements of the underlying representation to determine when basis vectors are annihilated. That is, as different from §5, we do not have to explicitly implement the annihilation rules dictated by the KIMC. This observation carries over into the following subsections.

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[13]We might just as well use $ELGS$, the full set of even lowering generators. The tradeoff is that whilst the coproducts of the nonsimple generators are more complex, there should be less levels to calculate.

6.4.1 Illustration: $\mathfrak{B}_1^0$ for $U_q[gl(3|1)]$

We illustrate the process by constructing the 3 dimensional $\mathfrak{B}_1^0$ for $U_q[gl(3|1)]$. Recall that we determined:

$$\left|\Psi_1^1\right\rangle = \overline{C}_0^{\frac{1}{2}}\left(q^{\frac{\alpha}{2}}\left|2\otimes1\right\rangle - \overline{q}^{\frac{\alpha}{2}}\left|1\otimes2\right\rangle\right), \tag{76}$$

hence $L_0 = \{\overline{C}_0^{\frac{1}{2}}\left(q^{\frac{\alpha}{2}}\left|2\otimes1\right\rangle - \overline{q}^{\frac{\alpha}{2}}\left|1\otimes2\right\rangle\right)\}$. We also have $ESLGS = \left\{E^2{}_1, E^3{}_2\right\}$, for which, from (25a), we have the coproducts:

$$\left.\begin{array}{rcl}\Delta(E^2{}_1) & = & (E^2{}_1 \otimes K_2^{\frac{1}{2}}\overline{K}_1^{\frac{1}{2}}) + (\overline{K}_2^{\frac{1}{2}}K_1^{\frac{1}{2}} \otimes E^2{}_1) \\ \Delta(E^3{}_2) & = & (E^3{}_2 \otimes K_3^{\frac{1}{2}}\overline{K}_2^{\frac{1}{2}}) + (\overline{K}_3^{\frac{1}{2}}K_2^{\frac{1}{2}} \otimes E^3{}_2).\end{array}\right\} \tag{77}$$

Applying the operators of (77) to (76), we obtain two vectors:

$$\left\{0, \overline{C}_0^{\frac{1}{2}}\left(\overline{q}^{\frac{\alpha}{2}}\left|1\otimes3\right\rangle - q^{\frac{\alpha}{2}}\left|3\otimes1\right\rangle\right)\right\}.$$

We discard the 0, and find that the second vector is already normalised and also not found in L_0, so we have $L_1 = \{\overline{C}_0^{\frac{1}{2}}\left(\overline{q}^{\frac{\alpha}{2}}\left|1\otimes3\right\rangle - q^{\frac{\alpha}{2}}\left|3\otimes1\right\rangle\right)\}$. As the components of L_0 and L_1 are independent, and indeed ordered, we have, at this stage:

$$L = \{\overline{C}_0^{\frac{1}{2}}\left(q^{\frac{\alpha}{2}}\left|2\otimes1\right\rangle - \overline{q}^{\frac{\alpha}{2}}\left|1\otimes2\right\rangle\right), \overline{C}_0^{\frac{1}{2}}\left(\overline{q}^{\frac{\alpha}{2}}\left|1\otimes3\right\rangle - q^{\frac{\alpha}{2}}\left|3\otimes1\right\rangle\right)\}.$$

Repeating this process on L_1, we find $L_2 = \left\{\overline{C}_0^{\frac{1}{2}}\left(q^{\frac{\alpha}{2}}\left|4\otimes1\right\rangle - \overline{q}^{\frac{\alpha}{2}}\left|1\otimes4\right\rangle\right)\right\}$. As before, we normalise (again already OK) and install this vector in its rightful position in our collection L, checking first to see if we have already met it. We thus have, after ordering:

$$L = \left\{\begin{array}{l}\overline{C}_0^{\frac{1}{2}}\left(q^{\frac{\alpha}{2}}\left|2\otimes1\right\rangle - \overline{q}^{\frac{\alpha}{2}}\left|1\otimes2\right\rangle\right) \\ \overline{C}_0^{\frac{1}{2}}\left(\overline{q}^{\frac{\alpha}{2}}\left|1\otimes3\right\rangle - q^{\frac{\alpha}{2}}\left|3\otimes1\right\rangle\right) \\ \overline{C}_0^{\frac{1}{2}}\left(q^{\frac{\alpha}{2}}\left|4\otimes1\right\rangle - \overline{q}^{\frac{\alpha}{2}}\left|1\otimes4\right\rangle\right)\end{array}\right\}.$$

Repeating again, we discover that $L_3 = \emptyset$, so the process is completed. Our orthonormal basis $\mathfrak{B}_1^0$ is thus the above L.

Note that, apart from some factoring, we have left this basis in the raw form that our MATHEMATICA code yields. The human calculator, preferring symmetry, may wish to multiply some of the vectors by (the phase factor) -1, but this is unnecessary for our purposes. In Appendix B, where our results are summarised, we have make some judicious changes of this nature for readability.

6.5 A nonorthogonal basis $\overline{\mathfrak{B}}_k$ for V_k

At this stage, we have established the orthonormal basis $\mathfrak{B}_k^0$ for V_k^0, and we wish to extend $\mathfrak{B}_k^0$ to a basis $\mathfrak{B}_k$ for V_k. This process involves two stages:

- Firstly, we use the KIMC to construct a basis $\overline{\mathfrak{B}}_k$ for V_k by the repeated actions of the odd lowering generators on the basis $\mathfrak{B}_k^0$, normalising and casting out repeated vectors as we go. This part of this process is detailed in this subsection: its appearance is similar to that of §6.4.

- Unlike $\mathfrak{B}_k^0$ however, the vectors of $\overline{\mathfrak{B}}_k$ are not guaranteed to be orthogonal, as V_k in general contains some weight multiplicities. That is, distinct vectors of the same weights will generally appear, and these are usually nonorthogonal. To deal with this problem, we apply a Gram–Schmidt process to orthonormalise $\overline{\mathfrak{B}}_k$ into the final $\mathfrak{B}_k$. To optimise this, we preprocess $\overline{\mathfrak{B}}_k$ by ordering its vectors by decreasing weight and then partitioning it into weight equivalence classes. We then need only apply the Gram–Schmidt process to each equivalence class. The end result is $\mathfrak{B}_k$, the desired orthonormal weight basis for V_k. This process is documented in §6.6.

Thus, we reproduce essentially the same algorithm as that used in §6.4, the essential differences being that the levels L_i are now $\mathbb{Z}$ graded levels of V_k, and that we act with $OLGS$ (the (full) set of odd lowering generators, see (45)), rather than with $ESLGS$.[14]

$$
\begin{aligned}
&i := 0 \\
&L_0 := \mathfrak{B}_k^0 \\
&\text{while } L_i \neq \emptyset \\
&\qquad L_{i+1} := (\Delta(OLGS) \cdot L_i) \setminus \{0\} \\
&\qquad \text{normalise } L_{i+1} \\
&\qquad L_{i+1} := L_{i+1} \setminus L \\
&\qquad L := L \cup L_{i+1} \\
&\qquad \text{sort } L \text{ by decreasing weight} \\
&\qquad \text{increment } i \\
&\overline{\mathfrak{B}}_k := L
\end{aligned}
$$

Again, the elements of $\mathfrak{B}_k$ are only unique up to phase factors, and our code doesn't select these, so the final results contain various factors of -1 that a human calculator would quickly purge. Furthermore, the weight ordering covers some elegant symmetries of the generators. In the results presented in the Appendix, we make some judicious cosmetic changes for readability.

---

[14]In this case, if we try to only use $OSLGS$, the set of *simple* odd lowering generators, then we miss some of the vectors of each level, which are obtained by nonsimple odd lowering generators, i.e. products of simple odd lowering generators with even lowering generators. The combination of the use $OSLGS$ and $ELGS$ (or, the repeated use of $ESLGS$) to build each level would require more calculations.

6.5.1 Illustration: $\overline{\mathfrak{B}}_1$ for $U_q[gl(3|1)]$

We illustrate the results using $\overline{\mathfrak{B}}_1$ for $U_q[gl(3|1)]$, continuing the example of §6.4.1. Here, $\overline{\mathfrak{B}}_1$ has 24 elements, sorted into equivalence classes of decreasing weight, and judiciously factored. To save space, we have written $\nabla \triangleq q + \overline{q}$, and $\overline{\nabla} \triangleq \nabla^{-1}$.

$$\overline{C}_0^{\frac{1}{2}}\left(-\overline{q}^{\frac{\alpha}{2}}\,|1\otimes 2\rangle + q^{\frac{\alpha}{2}}\,|2\otimes 1\rangle\right)$$

$$-\,|2\otimes 2\rangle$$

$$\overline{C}_0^{\frac{1}{2}}\left(+\overline{q}^{\frac{\alpha}{2}}\,|1\otimes 3\rangle - q^{\frac{\alpha}{2}}\,|3\otimes 1\rangle\right)$$

$$\overline{C}_0^{\frac{1}{2}}\overline{A}_{2,1}^{\frac{1}{2}}\left(A_0^{\frac{1}{2}}(+\overline{q}^{\alpha+\frac{1}{2}}\,|2\otimes 3\rangle + q^{\alpha+\frac{1}{2}}\,|3\otimes 2\rangle) + A_1^{\frac{1}{2}}(+\,|1\otimes 5\rangle - |5\otimes 1\rangle)\right)$$

$$\overline{\nabla}^{\frac{1}{2}}\left(+q^{\frac{1}{2}}\,|2\otimes 3\rangle + \overline{q}^{\frac{1}{2}}\,|3\otimes 2\rangle\right)$$

$$\overline{C}_1^{\frac{1}{2}}\left(-q^{\frac{\alpha}{2}+\frac{1}{2}}\,|2\otimes 5\rangle + \overline{q}^{\frac{\alpha}{2}+\frac{1}{2}}\,|5\otimes 2\rangle\right)$$

$$-\,|3\otimes 3\rangle$$

$$\overline{C}_1^{\frac{1}{2}}\left(+q^{\frac{\alpha}{2}+\frac{1}{2}}\,|3\otimes 5\rangle - \overline{q}^{\frac{\alpha}{2}+\frac{1}{2}}\,|5\otimes 3\rangle\right)$$

$$\overline{C}_0^{\frac{1}{2}}\left(-\overline{q}^{\frac{\alpha}{2}}\,|1\otimes 4\rangle + q^{\frac{\alpha}{2}}\,|4\otimes 1\rangle\right)$$

$$\overline{C}_0^{\frac{1}{2}}\overline{A}_{2,1}^{\frac{1}{2}}\left(A_0^{\frac{1}{2}}(-\overline{q}^{\alpha+\frac{1}{2}}\,|2\otimes 4\rangle - q^{\alpha+\frac{1}{2}}\,|4\otimes 2\rangle) + A_1^{\frac{1}{2}}(-\,|1\otimes 6\rangle + |6\otimes 1\rangle)\right)$$

$$\overline{\nabla}^{\frac{1}{2}}\left(-q^{\frac{1}{2}}\,|2\otimes 4\rangle - \overline{q}^{\frac{1}{2}}\,|4\otimes 2\rangle\right)$$

$$\overline{C}_1^{\frac{1}{2}}\left(+q^{\frac{\alpha}{2}+\frac{1}{2}}\,|2\otimes 6\rangle - \overline{q}^{\frac{\alpha}{2}+\frac{1}{2}}\,|6\otimes 2\rangle\right)$$

$$\overline{C}_0^{\frac{1}{2}}\overline{A}_{2,1}^{\frac{1}{2}}\left(A_0^{\frac{1}{2}}(+\overline{q}^{\alpha+\frac{1}{2}}\,|3\otimes 4\rangle + q^{\alpha+\frac{1}{2}}\,|4\otimes 3\rangle) + A_1^{\frac{1}{2}}(+\,|1\otimes 7\rangle - |7\otimes 1\rangle)\right)$$

$$\overline{\nabla}^{\frac{1}{2}}\left(+q^{\frac{1}{2}}\,|3\otimes 4\rangle + \overline{q}^{\frac{1}{2}}\,|4\otimes 3\rangle\right)$$

$$\overline{C}_0^{\frac{1}{2}}\overline{C}_1^{\frac{1}{2}}\overline{A}_{2,1}^{\frac{1}{2}}\left(\begin{array}{l} +A_2^{\frac{1}{2}}\left(+q^{\frac{\alpha}{2}}\,|1\otimes 8\rangle - \overline{q}^{\frac{\alpha}{2}}\,|8\otimes 1\rangle\right) + A_0^{\frac{1}{2}}\left(+\overline{q}^{\frac{\alpha}{2}+1}\,|2\otimes 7\rangle - q^{\frac{\alpha}{2}+1}\,|7\otimes 2\rangle\right) \\[4pt] +A_0^{\frac{1}{2}}\left(-\overline{q}^{\frac{\alpha}{2}}\,|3\otimes 6\rangle + q^{\frac{\alpha}{2}}\,|6\otimes 3\rangle\right) + A_0^{\frac{1}{2}}\left(-q^{\frac{3\alpha}{2}+1}\,|4\otimes 5\rangle + \overline{q}^{\frac{3\alpha}{2}+1}\,|5\otimes 4\rangle\right) \end{array}\right)$$

$$\overline{\nabla}^{\frac{1}{2}}\overline{C}_1^{\frac{1}{2}}\left(-q^{\frac{\alpha}{2}+1}\,|3\otimes 6\rangle + \overline{q}^{\frac{\alpha}{2}+1}\,|6\otimes 3\rangle - q^{\frac{\alpha}{2}}\,|4\otimes 5\rangle + \overline{q}^{\frac{\alpha}{2}}\,|5\otimes 4\rangle\right)$$

$$\overline{\nabla}^{\frac{1}{2}}\overline{C}_1^{\frac{1}{2}}\left(-q^{\frac{\alpha}{2}+1}\,|2\otimes 7\rangle + \overline{q}^{\frac{\alpha}{2}+1}\,|7\otimes 2\rangle - q^{\frac{\alpha}{2}}\,|3\otimes 6\rangle + \overline{q}^{\frac{\alpha}{2}}\,|6\otimes 3\rangle\right)$$

$$\overline{C}_1^{\frac{1}{2}}\overline{A}_{2,3}^{\frac{1}{2}}\left(A_1^{\frac{1}{2}}(-\overline{q}^{\frac{1}{2}}\,|5\otimes 6\rangle + q^{\frac{1}{2}}\,|6\otimes 5\rangle) + A_2^{\frac{1}{2}}(+q^{\alpha+1}\,|2\otimes 8\rangle + \overline{q}^{\alpha+1}\,|8\otimes 2\rangle)\right)$$

$$\overline{C}_1^{\frac{1}{2}}\left(+q^{\frac{\alpha}{2}+\frac{1}{2}}\,|3\otimes 7\rangle - \overline{q}^{\frac{\alpha}{2}+\frac{1}{2}}\,|7\otimes 3\rangle\right)$$

$$\overline{C}_1^{\frac{1}{2}}\overline{A}_{2,3}^{\frac{1}{2}}\left(A_1^{\frac{1}{2}}(+\overline{q}^{\frac{1}{2}}\,|5\otimes 7\rangle - q^{\frac{1}{2}}\,|7\otimes 5\rangle) + A_2^{\frac{1}{2}}(-q^{\alpha+1}\,|3\otimes 8\rangle - \overline{q}^{\alpha+1}\,|8\otimes 3\rangle)\right)$$

$$-\,|4\otimes 4\rangle$$

$$\overline{C}_1^{\frac{1}{2}}\left(+q^{\frac{\alpha}{2}+\frac{1}{2}}\,|4\otimes 6\rangle - \overline{q}^{\frac{\alpha}{2}+\frac{1}{2}}\,|6\otimes 4\rangle\right)$$

$$\overline{C}_1^{\frac{1}{2}}\left(-q^{\frac{\alpha}{2}+\frac{1}{2}}\,|4\otimes 7\rangle + \overline{q}^{\frac{\alpha}{2}+\frac{1}{2}}\,|7\otimes 4\rangle\right)$$

$$\overline{C}_1^{\frac{1}{2}}\overline{A}_{2,3}^{\frac{1}{2}}\left(A_1^{\frac{1}{2}}(-\overline{q}^{\frac{1}{2}}\,|6\otimes 7\rangle + q^{\frac{1}{2}}\,|7\otimes 6\rangle) + A_2^{\frac{1}{2}}(+q^{\alpha+1}\,|4\otimes 8\rangle + \overline{q}^{\alpha+1}\,|8\otimes 4\rangle)\right)$$

6.6 Orthonormalising $\overline{\mathfrak{B}}_k$ into $\mathfrak{B}_k$

At this stage, the KIMC has yielded a nonorthogonal (although normalised!) q graded symmetric basis $\overline{\mathfrak{B}}_k$ for V_k. With a view to constructing the projector P_k onto V_k (see §7.1), we require dual bases for V_k and its dual V_k^*. From knowledge of $\overline{\mathfrak{B}}_k$ there are two obvious ways to construct these dual bases:

- We might continue to regard $\overline{\mathfrak{B}}_k$ as our basis for V_k, and construct a (nonorthogonal) dual tensor product basis by the inversion of an overlap (i.e. metric) matrix. This process is described for $\overline{\mathfrak{B}}_1$ for $U_q[gl(3|1)]$ in [13], although those authors don't actually implement it.

- Alternatively, we can orthonormalise $\overline{\mathfrak{B}}_k$ into $\mathfrak{B}_k$ using a Gram–Schmidt process. The dual basis $\mathfrak{B}_k^*$ is then naturally orthonormal, and indeed trivial to write down.

Implementation of both of these methods has shown that, apart from being substantially more efficient, the latter method yields more tractable and symmetric results, so we choose it. Not only is it more elegant, but happily, the Gram–Schmidt process also allows us to maintain the q graded symmetry of the basis vectors.

A basic principle in *numerical* computation is to never invert a matrix unless absolutely necessary, as the process is both computationally inefficient and (numerically) unstable. The same comment about computational inefficiency certainly holds for the inversion of symbolic matrices. More seriously, for our current purposes, the inversion can bog down altogether due to difficulties in the simplification of algebraic expressions; a feature we might call 'symbolic instability'.

6.6.1 Illustration: $\mathfrak{B}_1$ for $U_q[gl(3|1)]$

We illustrate the Gram–Schmidt process by continuing the example from §6.5.1, that of $\overline{\mathfrak{B}}_1$ for $U_q[gl(3|1)]$. Observe that the partitions of $\overline{\mathfrak{B}}_1$ include three of size 2 and one of size 3. To convert $\overline{\mathfrak{B}}_1$ to $\mathfrak{B}_1$, we must orthogonalise each of those partitions. To illustrate, for the largest partition, we obtain the 3 vectors:

$$
\left\{
\begin{array}{l}
\overline{C}_0^{\frac{1}{2}}\overline{C}_1^{\frac{1}{2}}\overline{A}_{2,1}^{\frac{1}{2}}\left(
\begin{array}{l}
A_2^{\frac{1}{2}}\left(+q^{\frac{\alpha}{2}}\,|1\otimes 8\rangle - \overline{q}^{\frac{\alpha}{2}}\,|8\otimes 1\rangle\right) + A_0^{\frac{1}{2}}\left(+\overline{q}^{\frac{\alpha}{2}+1}\,|2\otimes 7\rangle - q^{\frac{\alpha}{2}+1}\,|7\otimes 2\rangle\right)\\[4pt]
+A_0^{\frac{1}{2}}\left(-\overline{q}^{\frac{3\alpha}{2}}\,|3\otimes 6\rangle + q^{\frac{\alpha}{2}}\,|6\otimes 3\rangle\right) + A_0^{\frac{1}{2}}\left(-q^{\frac{3\alpha}{2}+1}\,|4\otimes 5\rangle + \overline{q}^{\frac{3\alpha}{2}+1}\,|5\otimes 4\rangle\right)
\end{array}
\right)\\[20pt]
\overline{C}_1\overline{A}_{2,1}^{\frac{1}{2}}\left(
\begin{array}{l}
A_0^{\frac{1}{2}}A_2^{\frac{1}{2}}\overline{A}_1^{\frac{1}{2}}\left(-q^{\frac{\alpha}{2}}\,|1\otimes 8\rangle + \overline{q}^{\frac{\alpha}{2}}\,|8\otimes 1\rangle\right) + A_0\overline{A}_1^{\frac{1}{2}}\left(-\overline{q}^{\frac{\alpha}{2}+1}\,|2\otimes 7\rangle + q^{\frac{\alpha}{2}+1}\,|7\otimes 2\rangle\right)\\[4pt]
+A_1^{\frac{1}{2}}\left(-q^{\frac{3\alpha}{2}+1}\,|3\otimes 6\rangle + \overline{q}^{\frac{3\alpha}{2}+1}\,|6\otimes 3\rangle\right) + A_1^{\frac{1}{2}}\left(-\overline{q}^{\frac{\alpha}{2}}\,|4\otimes 5\rangle + q^{\frac{\alpha}{2}}\,|5\otimes 4\rangle\right)
\end{array}
\right)\\[20pt]
\overline{C}_1\overline{A}_{2,3}^{\frac{1}{2}}\left(
\begin{array}{l}
A_0^{\frac{1}{2}}A_2^{\frac{1}{2}}\overline{A}_1^{\frac{1}{2}}\left(+q^{\frac{\alpha}{2}}\,|1\otimes 8\rangle - \overline{q}^{\frac{\alpha}{2}}\,|8\otimes 1\rangle\right) + A_2\overline{A}_1^{\frac{1}{2}}\left(-q^{\frac{3\alpha}{2}+1}\,|2\otimes 7\rangle + \overline{q}^{\frac{3\alpha}{2}+1}\,|7\otimes 2\rangle\right)\\[4pt]
+A_1^{\frac{1}{2}}\left(-\overline{q}^{\frac{\alpha}{2}+1}\,|3\otimes 6\rangle + q^{\frac{\alpha}{2}+1}\,|6\otimes 3\rangle\right) + A_1^{\frac{1}{2}}\left(+\overline{q}^{\frac{\alpha}{2}}\,|4\otimes 5\rangle - q^{\frac{\alpha}{2}}\,|5\otimes 4\rangle\right)
\end{array}
\right)
\end{array}
\right\}.
$$

Observe that, again, our MATHEMATICA code has done some nontrivial work in simplifying the algebraic expressions in q and α. This work would present a significant barrier for a human calculator.

7 Projectors and R matrices for $V \otimes V$

7.1 Projectors P_k onto the V_k

At this stage, for each of the submodules $V_k \subset V \otimes V$, we have an orthonormal basis $\mathfrak{B}_k = \left\{ \left| \Psi_l^k \right\rangle \right\}_{l=1}^{\dim(V_k)}$ and, for each of their duals V_k^*, a corresponding dual basis $\mathfrak{B}_k^* = \left\{ \left\langle \Psi_l^k \right| \right\}_{l=1}^{\dim(V_k)}$, where $\left\langle \Psi_l^k \right| \triangleq \left| \Psi_l^k \right\rangle^\dagger$. Using these dual bases, it is a simple matter to construct the projectors $P_k : V \otimes V \to V_k$:

$$P_k = \sum_{j=1}^{\dim(V_k)} \left| \Psi_l^k \right\rangle \left\langle \Psi_l^k \right| ;$$

note that we must use (62) for the multiplication of tensor products.

We now make a change of notation. As we did for the matrix elements in §5.2, we replace $|i\rangle\langle j|$ with the elementary matrix $e_j^i \in M_{2^{mn}}$. We then use the notation $e_{jl}^{ik} \in M_{2^{2mn}}$ to indicate the two dimensional matrix form of the usual elementary rank 4 tensor $e_j^i \otimes e_l^k$, obtained by inserting a copy of e_l^k at each location of e_j^i. We find that the P_k are in general quite sparse, that is, only a small fraction of their 2^{4mn} components are nonzero.

7.1.1 Illustration: P_0 for $U_q[gl(3|1)]$

We illustrate using P_0 for the case $U_q[gl(3|1)]$, which has 125 (out of $2^{12} = 4096$) nonzero components. We present these components below, using horizontal lines to separate equivalence classes of symmetry.

$$\left\{ e_{11}^{11} \right\}$$

$$\frac{1}{C_0} \left\{ \begin{matrix} q^\alpha \left\{ e_{12}^{12}, e_{13}^{13}, e_{14}^{14} \right\} \\ \overline{q}^\alpha \left\{ e_{21}^{21}, e_{31}^{31}, e_{41}^{41} \right\} \end{matrix} \right\}, \quad \frac{A_1}{C_0 A_{2,1}} \left\{ \begin{matrix} q^{2\alpha} \left\{ e_{15}^{15}, e_{16}^{16}, e_{17}^{17} \right\} \\ \overline{q}^{2\alpha} \left\{ e_{51}^{51}, e_{61}^{61}, e_{71}^{71} \right\} \end{matrix} \right\}, \quad \frac{A_2}{C_0 C_1 A_{2,1}} \left\{ \begin{matrix} q^{3\alpha} \left\{ e_{18}^{18} \right\} \\ \overline{q}^{3\alpha} \left\{ e_{81}^{81} \right\} \end{matrix} \right\}$$

$$\frac{A_0}{C_0 A_{2,1}} \left\{ \begin{matrix} \overline{q} \left\{ e_{23}^{23}, e_{24}^{24}, e_{34}^{34} \right\} \\ q \left\{ e_{32}^{32}, e_{42}^{42}, e_{43}^{43} \right\} \end{matrix} \right\}, \quad \frac{A_0}{C_0 C_1 A_{2,1}} \left\{ \begin{matrix} q^\alpha \left\{ \overline{q}^2 e_{27}^{27}, e_{36}^{36}, q^2 e_{45}^{45} \right\} \\ \overline{q}^\alpha \left\{ q^2 e_{72}^{72}, e_{63}^{63}, \overline{q}^2 e_{54}^{54} \right\} \end{matrix} \right\}$$

$$\frac{1}{C_0} \left\{ \begin{matrix} - \left\{ e_{21}^{12}, e_{31}^{13}, e_{41}^{14} \right\} \\ + \left\{ e_{12}^{21}, e_{13}^{31}, e_{14}^{41} \right\} \end{matrix} \right\}, \quad \frac{A_1}{C_0 A_{2,1}} \left\{ \begin{matrix} e_{51}^{15}, e_{61}^{16}, e_{71}^{17} \\ e_{15}^{51}, e_{16}^{61}, e_{17}^{71} \end{matrix} \right\}, \quad \frac{A_2}{C_0 C_1 A_{2,1}} \left\{ \begin{matrix} -e_{81}^{18} \\ +e_{18}^{81} \end{matrix} \right\}$$

$$-\frac{A_0}{C_0 A_{2,1}} \left\{ \begin{matrix} e_{32}^{23}, e_{42}^{24}, e_{43}^{34} \\ e_{24}^{42}, e_{23}^{32}, e_{34}^{43} \end{matrix} \right\}, \quad \frac{A_0}{C_0 C_1 A_{2,1}} \left\{ \begin{matrix} + \left\{ e_{72}^{27}, e_{63}^{36}, e_{54}^{45} \right\} \\ - \left\{ e_{27}^{72}, e_{36}^{63}, e_{45}^{54} \right\} \end{matrix} \right\}$$

$$\frac{A_0^{\frac{1}{2}} A_1^{\frac{1}{2}}}{C_0 A_{2,1}} \left\{ \begin{matrix} q^{\alpha - \frac{1}{2}} \left\{ \begin{matrix} - \left\{ e_{23}^{15}, e_{24}^{16}, e_{34}^{17} \right\} \\ + \left\{ e_{15}^{23}, e_{16}^{24}, e_{17}^{34} \right\} \end{matrix} \right\}, & \overline{q}^{\alpha - \frac{1}{2}} \left\{ \begin{matrix} + \left\{ e_{32}^{51}, e_{42}^{61}, e_{43}^{71} \right\} \\ - \left\{ e_{51}^{32}, e_{61}^{42}, e_{71}^{43} \right\} \end{matrix} \right\} \\ q^{\alpha + \frac{1}{2}} \left\{ \begin{matrix} + \left\{ e_{32}^{15}, e_{42}^{16}, e_{43}^{17} \right\} \\ - \left\{ e_{15}^{32}, e_{16}^{42}, e_{17}^{43} \right\} \end{matrix} \right\}, & \overline{q}^{\alpha + \frac{1}{2}} \left\{ \begin{matrix} - \left\{ e_{23}^{51}, e_{24}^{61}, e_{34}^{71} \right\} \\ + \left\{ e_{51}^{23}, e_{61}^{24}, e_{71}^{34} \right\} \end{matrix} \right\} \end{matrix} \right\}$$

$$\frac{A_0^{\frac{1}{2}} A_2^{\frac{1}{2}}}{C_0 C_1 A_{2,1}} \left\{ \begin{matrix} q^{2\alpha} \left\{ \begin{matrix} -\overline{q} e_{27}^{18}, +e_{36}^{18}, -q e_{45}^{18} \\ +\overline{q} e_{18}^{27}, -e_{18}^{36}, +q e_{18}^{45} \end{matrix} \right\}, & \overline{q}^{2\alpha} \left\{ \begin{matrix} +q e_{72}^{81}, -e_{63}^{81}, +\overline{q} e_{54}^{81} \\ -q e_{81}^{72}, +e_{81}^{63}, -\overline{q} e_{81}^{54} \end{matrix} \right\} \\ q^\alpha \left\{ \begin{matrix} +q e_{72}^{18}, -e_{63}^{18}, +\overline{q} e_{54}^{18} \\ +q e_{18}^{72}, -e_{18}^{63}, +\overline{q} e_{18}^{54} \end{matrix} \right\}, & \overline{q}^\alpha \left\{ \begin{matrix} +\overline{q} e_{27}^{81}, -e_{36}^{81}, +q e_{45}^{81} \\ +\overline{q} e_{81}^{27}, -e_{81}^{36}, +q e_{81}^{45} \end{matrix} \right\} \end{matrix} \right\}$$

$$\frac{A_0}{C_0 C_1 A_{2,1}} \left\{ \begin{matrix} q^\alpha \left\{ \begin{matrix} -\overline{q} e_{36}^{27}, +e_{45}^{27}, -q e_{45}^{36} \\ -\overline{q} e_{27}^{36}, +e_{27}^{45}, -q e_{36}^{45} \end{matrix} \right\}, & \overline{q}^\alpha \left\{ \begin{matrix} -q e_{63}^{72}, +e_{54}^{72}, -\overline{q} e_{54}^{63} \\ -q e_{72}^{63}, +e_{72}^{54}, -\overline{q} e_{63}^{54} \end{matrix} \right\} \\ q \left\{ \begin{matrix} +e_{36}^{72}, -q e_{45}^{72}, +e_{45}^{63} \\ -e_{72}^{36}, +q e_{72}^{45}, -e_{63}^{45} \end{matrix} \right\}, & \overline{q} \left\{ \begin{matrix} -e_{63}^{27}, +\overline{q} e_{54}^{27}, -e_{54}^{36} \\ +e_{27}^{63}, -\overline{q} e_{27}^{54}, +e_{36}^{54} \end{matrix} \right\} \end{matrix} \right\}.$$

7.2 R matrices $\check{R}^{m,n}(u)$ and $\check{R}^{m,n}$

We may now form the trigonometric R matrix $\check{R}^{m,n}(u)$ as a weighted sum of the projectors, where the weights are the eigenvalues of $\check{R}^{m,n}(u)$ on the submodules. For the *special* case of our representations labeled $\Lambda = (\dot{0}_m|\dot{\alpha}_n)$, these eigenvalues are actually *known* [11]. The quantum R matrix is then the spectral limit $\check{R}^{m,n} = \lim_{u\to\infty} \check{R}^{m,n}(u)$. Again, what follows here strictly applies only to the $m \geqslant n$ case, but given the natural isomorphism between $U_q[gl(m|n)]$ and $U_q[gl(n|m)]$, this is unimportant.

Again using the notation introduced in §6, especially noting the definition of λ_γ in (64), we have the following expression for $\check{R}^{m,n}(u)$, normalised such that its 'first' component (i.e. the coefficient of e_{11}^{11}) is unity (for applications, other normalisations may be applicable, e.g. see [7]):

$$\check{R}^{m,n}(u) = \sum_{[\gamma]} \Xi_{2\Lambda+\lambda_\gamma}(u) P_{2\Lambda+\lambda_\gamma}, \tag{78}$$

where, again, as in (65), the sum is over all allowable Young diagrams $[\gamma]$ and $P_{2\Lambda+\lambda_\gamma}$ is the projector onto the submodule $V_{2\Lambda+\lambda_\gamma}$. Recalling that we intend $\gamma = \gamma_1, \gamma_2, \ldots, \gamma_r$, the eigenvalue $\Xi_{2\Lambda+\lambda_\gamma}(u)$ is:

$$\Xi_{2\Lambda+\lambda_\gamma}(u) = \prod_{j=1}^{r} \prod_{i=1}^{\gamma_j} \frac{[\alpha+j-i+u]_q}{[\alpha+j-i-u]_q}, \tag{79}$$

where, for the empty Young diagram case, we intend $\Xi_{2\Lambda}(u) = 1$. Note that we have substituted $x = q^{-2u}$ from the original x used in the *multiplicative* Yang–Baxter equations of [11]: *our* Yang–Baxter equations are *additive* in variable u (see §7.3).

Then, as $\check{R}^{m,n}$ is the spectral limit $\lim_{u\to\infty} \check{R}^{m,n}(u)$, we have:

$$\check{R}^{m,n} = \sum_{[\gamma]} \xi_{2\Lambda+\lambda_\gamma} P_{2\Lambda+\lambda_\gamma}, \tag{80}$$

where $\xi_{2\Lambda+\lambda_\gamma}(u) \triangleq \lim_{u\to\infty} \Xi_{2\Lambda+\lambda_\gamma}(u)$. Again, the coefficient of e_{11}^{11} is unity. Evaluating this limit:

$$\lim_{u\to\infty} \Xi_{2\Lambda+\lambda_\gamma}(u) = \prod_{j=1}^{r} \prod_{i=1}^{\gamma_j} (-)q^{2(\alpha+j-i)} = \prod_{j=1}^{r} (-)^{\gamma_j} q^{\gamma_j(2\alpha+2j-\gamma_j-1)},$$

where we have applied the observation: $\lim_{u\to\infty} \frac{[X+u]_q}{[X-u]_q} = -q^{+2X}$. Strictly, this requires that $|q| > 1$, and this is perhaps not so sensible, as q is in some sense, a *deformation* parameter, so should be *small*. If, instead, we assume that $|q| < 1$, then the limit becomes $-q^{-2X}$, and the expressions for $\xi_{2\Lambda+\lambda_\gamma}$ both above and below (in (81)), remain valid under the mapping $q \mapsto \overline{q}$. In the final analysis, this simply means that we obtain quantum R matrices related by $q \mapsto \overline{q}$. As above, we are considering *generic* q (i.e. q is not a root of unity), and to ensure that, we can demand $|q| \neq 1$.

These considerations aside, we obtain:

$$\xi_{2\Lambda+\lambda_\gamma} = (-)^N q^{\sum_{j=1}^{r} \gamma_j (2\alpha+2j-\gamma_j-1)}, \tag{81}$$

where $N = \sum_{j=1}^{r} \gamma_j$ is the $\mathbb{Z}$ graded level of $V_{2\Lambda+\lambda_\gamma}$ (cf. §6.2). Note that we intend $\xi_{2\Lambda} = 1$, in agreement with $\Xi_{2\Lambda}(u) = 1$.

Thus, we have two methods to compute $\check{R}^{m,n}$. Firstly, we may explicitly evaluate it as the spectral limit of $\check{R}^{m,n}(u)$, itself computed by substituting (79) into (78). Secondly, we may directly substitute (81) into (80), bypassing the construction of $\check{R}^{m,n}(u)$ altogether. This method is of course less computationally expensive, so in practice, we use it, but we have also implemented the former method, which is useful for checking consistency.

7.2.1 Illustration: The R matrix decompositions for $U_q[gl(m|1)]$

Again resurrecting the abusive notion of (66), we may replace the explicit (78) and (80) with:

$$\check{R}^{m,n}(u) = \sum_k \Xi_k(u) P_k, \qquad \check{R}^{m,n} = \sum_k \xi_k P_k.$$

For the special case $n = 1$, which we are interested in for physical applications, these simplify to:

$$\check{R}^{m,1}(u) = \sum_{k=0}^{m} \Xi_k(u) P_k, \qquad \check{R}^{m,1} = \sum_{k=0}^{m} \xi_k P_k,$$

where $\Xi_k(u)$ and ξ_k are the following eigenvalues ([15], and cf. [13]):

$$\Xi_k(u) = \prod_{j=0}^{k-1} \frac{[\alpha+j+u]_q}{[\alpha+j-u]_q}, \qquad \xi_k = (-)^k q^{k(2\alpha+k-1)};$$

here we of course again intend $\Xi_0(u) = \xi_0 = 1$.

7.3 Yang–Baxter equations

To be certain, $\check{R}^{m,n}(u)$ satisfies the following graded version of the (additive) trigonometric Yang–Baxter equation (TYBE):

$$(-)^{[b'][c']+[a'][c]+[a][b]+[b'][b'']} \check{R}(u)^{c''b''}_{b'c'} \check{R}(u+v)^{c'a''}_{a'c} \check{R}(v)^{b'a'}_{ab}$$
$$= (-)^{[a'][b']+[a][c']+[b][c]+[b][b']} \check{R}(v)^{b''a''}_{a'b'} \check{R}(u+v)^{c''a'}_{ac'} \check{R}(u)^{c'b'}_{bc}, \qquad (82)$$

where we have written $[a]$ for $[|a\rangle]$. The parity factors in (82) may be removed by the following transformation (e.g. see [4]):

$$\check{R}^{a'b'}_{ab}(u) \mapsto (-)^{[a]([b]+[b'])} \check{R}^{a'b'}_{ab}(u),$$

after which $\check{R}(u)$ which satisfied (82) now satisfies the usual ungraded TYBE:

$$\check{R}(u)^{c''b''}_{b'c'} \check{R}(u+v)^{c'a''}_{a'c} \check{R}(v)^{b'a'}_{ab} = \check{R}(v)^{b''a''}_{a'b'} \check{R}(u+v)^{c''a'}_{ac'} \check{R}(u)^{c'b'}_{bc}, \qquad (83)$$

written in noncomponent form as:

$$\check{R}_{12}(u)\check{R}_{23}(u+v)\check{R}_{12}(v) = \check{R}_{23}(v)\check{R}_{12}(u+v)\check{R}_{23}(u). \qquad (84)$$

In the spectral limit $\check{R} = \lim_{u\to\infty} \check{R}(u)$, this of course becomes a quantum Yang–Baxter equation (QYBE):

$$\check{R}_{12}\check{R}_{23}\check{R}_{12} = \check{R}_{23}\check{R}_{12}\check{R}_{23}, \qquad (85)$$

viz $(\check{R}\otimes I)(I\otimes \check{R})(\check{R}\otimes I) = (I\otimes \check{R})(\check{R}\otimes I)(I\otimes \check{R})$, familiar as the braid relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$.

Defining $R(u) \triangleq P\check{R}(u)$, where P is a permutation operator, yields a trigonometric R matrix $R(u)$ satisfying the following version of (84):

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u). \qquad (86)$$

This transformation amounts to the mapping: $R(u)^{a'b'}_{ab} = \check{R}(u)^{b'a'}_{ab}$. In component form, (86) is more symmetric than (83):

$$R(u)^{b''c''}_{b'c'} R(u+v)^{a''c'}_{a'c} R(v)^{a'b'}_{ab} = R(v)^{a''b''}_{a'b'} R(u+v)^{a'c''}_{ac'} R(u)^{b'c'}_{bc}.$$

7.4 An alternative construction of the quantum R matrix

In §6 and §7, we described the construction of trigonometric and quantum R matrices corresponding to the tensor products of representations of highest weight Λ, from explicit knowledge of the decomposition of the tensor product, and the eigenvalues of the R matrices on the subspaces of the decomposition. This method is limited to such situations where this data is *known*.

An alternative approach to the construction of *quantum* R matrices sidesteps the construction of bases $\mathfrak{B}_k$ and projectors P_k altogether, instead using only the knowledge of the matrix elements. That is, say that we know the 'universal' (i.e. algebraic) form of the R matrix, i.e.:

$$\mathcal{R} = \sum_i a_i \otimes b_i, \tag{87}$$

where $a_i, b_i \in U_q[gl(m|n)]$, for some hopefully finite sum over indices i. Then, for any particular representation of highest weight Λ, we may obtain a quantum R matrix $\pi_{\Lambda \otimes \Lambda}(\mathcal{R})$ satisfying a parameter-free version of (86) from $\mathcal{R}$, by simply replacing the a_i and b_i with their matrix representations. In fact, for $U_q[gl(m|n)]$, in [19] we find formulae for $\mathcal{R}$ of the form (87), so this method is feasible.

Implementation of this approach has significant advantages over the current method; apart from being simpler, and greatly reducing computational effort, it is considerably more general in that it does not require knowledge of the tensor product decomposition.

A substantial loss is that is does not yield explicit *trigonometric* R matrices, so we cannot use this method for *physical* applications – recall that our primary intended application is topological, being the construction of link invariants, for which we only require *quantum* R matrices. However, there is a further loss. An interesting alternative approach to constructing quantum R matrices involves constructing families of distinct but 'gauge equivalent' quantum R matrices, starting from a single trigonometric R matrix. (Details of investigations into this for the $U_q[gl(2|1)]$ case appear in [20].) Clearly, we also lose this alternative approach if we can't construct trigonometric R matrices. Furthermore, the method described in the present work incorporates the foreknowledge of the eigenvalues of the quantum and trigonometric R matrices. This knowledge is special to our representations, and it may be be used to assist analysis of the associated link invariants (again, see [20]). In the alternative construction, we have no such knowledge in general.

The only barrier to implementing this method is that the results of [19] are presented in a somewhat abstract form which would require considerable modification before being directly useful in the framework described herein. This approach is outside the scope of this paper; it is left as a future project.

8 Implementation and results

The entire process has been implemented as a suite of functions in the interpreted environment of MATHEMATICA. The procedure to construct the R matrices requires several stages (viz the algorithms of §4 to §7), and the several thousand lines of MATHEMATICA code are broken down into functional units to achieve this. The code is available on request from the author.

8.1 Data structure for the $U_q[gl(m|n)]$ generators

A challenging issue in implementation is to find a consistent data structure to represent the algebra generators. This problem arises as the Cartan generators K_a often appear exponentiated as K_a^N, where N is not necessarily a positive integer. Whilst the unexponentiated K_a (equivalently the $E^a{}_a$) *are* a basis for the Cartan subalgebra, if we use them as our data structures, then we must deal with the problem of how to express and manipulate their exponentials. Thus it proves pragmatic to regard the more general 'generators' K_a^N as the logical units for computation. Beyond this, our data structure must uniformly incorporate the non-Cartan $E^a{}_b$, which are never exponentiated. The following MATHEMATICA pattern integrates these two disparate expressions into a coherent form:

$$\texttt{Generator[Uqgl[m, n], a, b, N]},$$

where `m` and `n` are the fixed m and n defining the algebra, and, if $\texttt{a} \neq \texttt{b}$ (and `N` is fixed as 1), then we intend $E^a{}_b$, and if $\texttt{a} = \texttt{b}$, then we intend K_a^N.

Using this pattern, we are able to implement the collection of PBW commutators of §4.2 in only a few hundred lines of code.

8.2 General comments

Beyond the data structure, the following aspects of the code are specifically interesting as expositions of the use of MATHEMATICA.

1. Implementation of the PBW commutators to normal order strings of generators uses the repeated application of `Rules` to find a fixed point.

2. The code establishes and solves a system of linear equations to determine the parameters defining the highest weight vectors $\left|\Psi_1^k\right\rangle$ of the V_k, where even the size of this system is not specified in advance (see §6.3). Doing this manually is particularly finicky due to the semantic complexity of the expressions involved. (In the literature, this process is specifically *avoided* whenever possible; e.g. [13] contains two different kludges.)

3. Throughout, it has proven possible to maintain tensor product vectors and rank 4 tensor components in q graded symmetric combinations, which facilitates both the simplification of expressions and the presentation of the output. This has been achieved by applying rewrite rules to nonsymmetric expressions in a carefully controlled manner.

8.3   Limitations

The computations are computationally inefficient, and this is due mostly to the fact that the algorithms used are direct, and not refined. Although there are no theoretical limits to m and n, computer storage and human patience mean that a current reasonable practical limit is $mn \leqslant 4$. Whilst the technically difficult translation of the interpreted MATHEMATICA code into a compiled language would increase the speed of the computations enormously, storage requirements would still limit mn to perhaps 7 in the general case.

8.4   Summary of results

Both $\check{R}^{m,1}(u)$ and $\check{R}^{m,1}$ have been obtained for $m = 1, 2, 3, 4$. Fixing m, these are rank 4 tensors, where the tensor indices run from 1 to 2^m (i.e. the dimension of the underlying representation). Thus, they contain $D_m \triangleq 2^{4m}$ (albeit mostly zero) components. Where N'_m and N_m denote the number of nonzero components of $\check{R}^{m,1}(u)$ and $\check{R}^{m,1}$ respectively; as $\check{R}^{m,1}$ is the spectral limit of $\check{R}^{m,1}(u)$, we find, as expected, that $N_m \leqslant N'_m$. We also find that $N'_m = 6^m$ (why?). Where $s_m \triangleq N_m/D_m$ denotes the sparsity of $\check{R}^{m,1}$, we find that s_m rapidly decreases with increasing m. Table 4 presents this data for $m = 1, 2, 3, 4$.

m	D_m	N'_m	N_m	$s_m(\%)$
1	16	6	5	31.3
2	256	36	26	10.2
3	4096	216	139	3.4
4	65536	1296	758	1.1

Table 4:   Data for R matrix construction.

For good measure, we also record the numbers of components of each of the $m+1$ projectors for the above cases, finding them similar to N'_m and N_m. This data is included in Table 5, which also records submodule dimensions.

m	submodule dimensions	projector sizes
1	$2, 2$	$5, 5$
2	$4, 8, 4$	$25, 34, 25$
3	$8, 24, 24, 8$	$125, 199, 199, 125$
4	$16, 64, 96, 64, 16$	$625, 1124, 1254, 1124, 625$

Table 5:   Dimensions of the submodules in the tensor product, and the number of nonzero components of the corresponding projectors.

Computer run times involved are listed in Table 6. These show a rapid increase in cost with increasing m. This is accompanied by an exorbitant increase in storage required.

	m			
	1	2	3	4
Representations	0.10	0.5	2.8	20
TP submodule bases	0.74	7.4	72.6	1559
TP projectors	0.12	4.0	55.4	877
$\check{R}^{m,1}(u)$	0.27	6.5	89.5	1736
$\check{R}^{m,1}$	0.04	1.4	18.3	570

Table 6: Run times for various calculations (in CPU seconds), using MATHE-MATICA 3 on a SUN ULTRA 60 computer. The timing for construction of the 5 submodule bases for $U_q[gl(4|1)]$ is the sum of the timings for the individual modules, viz: $1559 = 27 + 201 + 1061 + 218 + 53$.

Of these, the $U_q[gl(1|1)]$ case can be done by hand in a few hours [2]; the complete $U_q[gl(2|1)]$ case appears in my PhD thesis [4], and took several weeks to do by hand; partial details (i.e. up to calculation of 3 out of 4 of the $\overline{\mathfrak{B}}_k$) for the $U_q[gl(3|1)]$ case appear in [13]; whilst the $U_q[gl(4|1)]$ case is entirely new.

Of interest is that $\check{R}^{m,1}$ generally contains non-binomial irreducible polynomial factors as well as various q brackets.

By direct substitution, we have been able to verify that each $\check{R}^{m,1}(u)$ satisfies (84)[15] and that each $\check{R}^{m,1}$ satisfies (85).

Acknowledgements

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[15]We didn't check $\check{R}^{4,1}(u)$, as the computations would have been excessively expensive.

A Proofs of various lemmas

A.1 A commutation lemma – the proof of (15)

Lemma 2

$$K_a^N E^b{}_c = q_a^{N(\delta_b^a - \delta_c^a)} E^b{}_c K_a^N,$$

for any meaningful indices b, c, and any power N.

Proof:

Firstly, we show the following result:

$$K_a E^b{}_c = q_a^{(\delta_b^a - \delta_c^a)} E^b{}_c K_a, \tag{88}$$

for *any* meaningful indices $b \neq c$, not just for the simple generators where we have $|b - c| = 1$.

To see this, we first consider the case $b > c$, so that $E^b{}_c$ is a lowering generator. We use induction on $b - c$ to show the result, assuming (88) for some $b > c$, where we know that it is true for $c = b - 1$ by (14). Then:

$$
\begin{aligned}
K_a E^b{}_{c-1} \;\overset{(6a)}{=}\; & K_a \left(E^b{}_c E^c{}_{c-1} - q_c E^c{}_{c-1} E^b{}_c \right) \\
= & \left(K_a E^b{}_c \right) E^c{}_{c-1} - q_c \left(K_a E^c{}_{c-1} \right) E^b{}_c \\
\overset{(88)}{=} & q_a^{(\delta_b^a - \delta_c^a)} E^b{}_c \left(K_a E^c{}_{c-1} \right) - q_c q_a^{(\delta_c^a - \delta_{c-1}^a)} \left(E^c{}_{c-1} K_a \right) E^b{}_c \\
\overset{(14)}{=} & q_a^{(\delta_b^a - \delta_c^a)} E^b{}_c q_a^{(\delta_c^a - \delta_{c-1}^a)} E^c{}_{c-1} K_a - \\
& q_c q_a^{(\delta_c^a - \delta_{c-1}^a)} E^c{}_{c-1} q_a^{(\delta_b^a - \delta_c^a)} E^b{}_c K_a \\
= & q_a^{(\delta_b^a - \delta_c^a + \delta_c^a - \delta_{c-1}^a)} \left(E^b{}_c E^c{}_{c-1} - q_c E^c{}_{c-1} E^b{}_c \right) K_a \\
\overset{(6a)}{=} & q_a^{(\delta_b^a - \delta_{c-1}^a)} E^b{}_{c-1} K_a.
\end{aligned}
$$

Thus, (88) is true for $E^b{}_{c-1}$ if it is true for $E^b{}_c$, and as it is true for $c = b - 1$, thus it is true for all $b > c$. The proof for raising generators follows by a trivial analogy.

Next, observe that setting $b = c$ in (88) also yields a true statement. That is, (88) then states $K_a E^b{}_b = E^b{}_b K_a$, which is equivalent to (12) when we replace K_a with $q^{(-)^{[a]} E^a{}_a}$, and expand the exponential as a power series.

Finally, replacement of q with q^N in (88) and again using $K_a = q^{(-)^{[a]} E^a{}_a}$ allows us to write the more general statement in (15). □

Having proven this, in retrospect, it appears that some use of the 'generalised Lusztig automorphisms' [8, 28] might facilitate a more elegant proof of the recursion in (88).

A.2 A coproduct lemma – the proof of (28)

Lemma 3

$$\Delta(E^a{}_b) \;=\; E^a{}_b \otimes K_a^{\frac12 S^a_b}\overline{K}_b^{\frac12 S^a_b} + \overline{K}_a^{\frac12 S^a_b} K_b^{\frac12 S^a_b} \otimes E^a{}_b - $$
$$S^a_b \sum_c \Delta_c\left(\overline{K}_c^{\frac12 S^a_b} K_b^{\frac12 S^a_b} E^a{}_c \otimes E^c{}_b K_a^{\frac12 S^a_b}\overline{K}_c^{\frac12 S^a_b}\right),$$

for any *valid indices* a, b *(including* $a = b$*), where the sum ranges over all* c
strictly between a *and* b*, and is simply ignored if* $|a - b| \leqslant 1$.

Proof:

Firstly, note that, for simple generators i.e. $|a - b| \leqslant 1$, the result is just the coproduct of (27):

$$\Delta(E^a{}_b) = E^a{}_b \otimes K_a^{\frac12 S^a_b}\overline{K}_b^{\frac12 S^a_b} + \overline{K}_a^{\frac12 S^a_b} K_b^{\frac12 S^a_b} \otimes E^a{}_b,$$

as the sum is ignored. Specifically, it applies to the Cartan generators $a = b$, being $\Delta(E^a{}_a) = E^a{}_a \otimes \mathrm{Id} + \mathrm{Id} \otimes E^a{}_a$, which is equivalent to (25c) once we make the identification $K_a = q^{(-)^{[a]}E^a{}_a}$.

More generally, given (27), and the expansion of the nonsimple generators (6), the following straightforward induction on $|a - b|$ shows that the result follows for arbitrary *non*simple generators $E^a{}_b$.

We first deal with the lowering case $a > b$, viz $S^a_b = +1$. For the inductive step, we assume the truth of our statement for $a - b = p$, for some $p > 1$, and we show that this implies its truth for $a - b = p + 1$. Thus, beginning with:

$$\Delta\left(E^a{}_{b-1}\right) \stackrel{(6a)}{=} \Delta\left(E^a{}_b E^b{}_{b-1} - q_b E^b{}_{b-1} E^a{}_b\right)$$
$$\stackrel{(26)}{=} \Delta\left(E^a{}_b\right)\Delta\left(E^b{}_{b-1}\right) - q_b \Delta\left(E^b{}_{b-1}\right)\Delta\left(E^a{}_b\right),$$

and expanding $\Delta\left(E^a{}_b\right)$ using our inductive hypothesis and $\Delta\left(E^b{}_{b-1}\right)$ from the definition (25a), we obtain:

$$\Delta\left(E^a{}_{b-1}\right) =$$

$$\left\{
\begin{array}{ll}
+E^a{}_b E^b{}_{b-1} \otimes K_a^{\frac12}\overline{K}_b^{\frac12} K_b^{\frac12}\overline{K}_{b-1}^{\frac12} & -q_b E^b{}_{b-1} E^a{}_b \otimes K_b^{\frac12}\overline{K}_{b-1}^{\frac12} K_a^{\frac12}\overline{K}_{b-1}^{\frac12} \\
+\overline{K}_a^{\frac12} K_b^{\frac12}\overline{K}_b^{\frac12} K_{b-1}^{\frac12} \otimes E^a{}_b E^b{}_{b-1} & -q_b \overline{K}_b^{\frac12} K_{b-1}^{\frac12}\overline{K}_a^{\frac12} K_b^{\frac12} \otimes E^b{}_{b-1} E^a{}_b \\
+\overline{K}_a^{\frac12} K_b^{\frac12} E^b{}_{b-1} \otimes E^a{}_b K_b^{\frac12}\overline{K}_{b-1}^{\frac12} & -q_b E^b{}_{b-1}\overline{K}_a^{\frac12} K_b^{\frac12} \otimes K_b^{\frac12}\overline{K}_{b-1}^{\frac12} E^a{}_b \\
+E^a{}_b \overline{K}_b^{\frac12} K_{b-1}^{\frac12} \otimes K_a^{\frac12}\overline{K}_b^{\frac12} E^b{}_{b-1} & -q_b \overline{K}_b^{\frac12} K_{b-1}^{\frac12} E^a{}_b \otimes E^b{}_{b-1} K_a^{\frac12}\overline{K}_b^{\frac12}
\end{array}
\right\} -$$

$$\sum_{c=b+1}^{a-1}(q_c - \overline{q}_c)\left\{
\begin{array}{c}
+\overline{K}_c^{\frac12} K_b^{\frac12} E^a{}_c E^b{}_{b-1} \otimes E^c{}_b K_a^{\frac12}\overline{K}_c^{\frac12} K_b^{\frac12}\overline{K}_{b-1}^{\frac12} \\
-q_b E^b{}_{b-1}\overline{K}_c^{\frac12} K_b^{\frac12} E^a{}_c \otimes K_b^{\frac12}\overline{K}_{b-1}^{\frac12} E^c{}_b K_a^{\frac12}\overline{K}_c^{\frac12} \\
+\overline{K}_c^{\frac12} K_b^{\frac12} E^a{}_c \overline{K}_b^{\frac12} K_{b-1}^{\frac12} \otimes E^c{}_b K_a^{\frac12}\overline{K}_c^{\frac12} E^b{}_{b-1} \\
-q_b \overline{K}_b^{\frac12} K_{b-1}^{\frac12}\overline{K}_c^{\frac12} K_b^{\frac12} E^a{}_c \otimes E^b{}_{b-1} E^c{}_b K_a^{\frac12}\overline{K}_c^{\frac12}
\end{array}
\right\}$$

$$\triangleq (t_1 + t_2 + t_3 + t_4) - \sum_{c=b+1}^{a-1}\Delta_c(t_5 + t_6).$$

55

Note that in the above, no parity factors appear, as $E^a{}_b$ and $E^b{}_{b-1}$ cannot both simultaneously be odd. Next, we examine the terms t_i:

$$
\begin{aligned}
t_1 &= \left(E^a{}_b E^b{}_{b-1} - q_b E^b{}_{b-1} E^a{}_b\right) \otimes K_a^{\frac{1}{2}}\overline{K}_{b-1}^{\frac{1}{2}} \overset{(6a)}{=} E^a{}_{b-1} \otimes K_a^{\frac{1}{2}}\overline{K}_{b-1}^{\frac{1}{2}} \\
t_2 &= \overline{K}_a^{\frac{1}{2}}K_{b-1}^{\frac{1}{2}} \otimes \left(E^a{}_b E^b{}_{b-1} - q_b E^b{}_{b-1} E^a{}_b\right) \overset{(6a)}{=} \overline{K}_a^{\frac{1}{2}}K_{b-1}^{\frac{1}{2}} \otimes E^a{}_{b-1} \\
t_3 &\overset{(15)}{=} \left(\overline{K}_a^{\frac{1}{2}}K_b^{\frac{1}{2}}E^b{}_{b-1} \otimes E^a{}_b K_b^{\frac{1}{2}}\overline{K}_{b-1}^{\frac{1}{2}}\right)\left(1 - q_b \overline{q}_b^{\frac{1}{2}}\overline{q}_b^{\frac{1}{2}}\right) = 0 \\
t_4 &\overset{(15)}{=} \left(\overline{K}_b^{\frac{1}{2}}K_{b-1}^{\frac{1}{2}}E^a{}_b \otimes E^b{}_{b-1} K_a^{\frac{1}{2}}\overline{K}_b^{\frac{1}{2}}\right)\left(\overline{q}_b^{\frac{1}{2}}\overline{q}_b^{\frac{1}{2}} - q_b\right) \\
&= -\Delta_b \left(\overline{K}_b^{\frac{1}{2}}K_{b-1}^{\frac{1}{2}}E^a{}_b \otimes E^b{}_{b-1} K_a^{\frac{1}{2}}\overline{K}_b^{\frac{1}{2}}\right) \\
t_5 &\overset{(15)}{=} \left(\overline{K}_c^{\frac{1}{2}}K_b^{\frac{1}{2}}E^a{}_c E^b{}_{b-1} \otimes E^c{}_b K_a^{\frac{1}{2}}\overline{K}_c^{\frac{1}{2}}K_b^{\frac{1}{2}}\overline{K}_{b-1}^{\frac{1}{2}}\right)\left(1 - q_b \overline{q}_b^{\frac{1}{2}}\overline{q}_b^{\frac{1}{2}}\right) = 0 \\
t_6 &\overset{(15)}{=} \overline{K}_c^{\frac{1}{2}}K_{b-1}^{\frac{1}{2}}E^a{}_c \otimes \left(E^c{}_b E^b{}_{b-1} - q_b E^b{}_{b-1}E^c{}_b\right)K_a^{\frac{1}{2}}\overline{K}_c^{\frac{1}{2}} \\
&\overset{(6a)}{=} \overline{K}_c^{\frac{1}{2}}K_{b-1}^{\frac{1}{2}}E^a{}_c \otimes E^c{}_{b-1}K_a^{\frac{1}{2}}\overline{K}_c^{\frac{1}{2}}.
\end{aligned}
$$

Thus, we have:

$$
\begin{aligned}
\Delta(E^a{}_{b-1}) &= E^a{}_{b-1} \otimes K_a^{\frac{1}{2}}\overline{K}_{b-1}^{\frac{1}{2}} + \overline{K}_a^{\frac{1}{2}}K_{b-1}^{\frac{1}{2}} \otimes E^a{}_{b-1} - \\
&\qquad \Delta_b \left(\overline{K}_b^{\frac{1}{2}}K_{b-1}^{\frac{1}{2}}E^a{}_b \otimes E^b{}_{b-1} K_a^{\frac{1}{2}}\overline{K}_b^{\frac{1}{2}}\right) - \\
&\qquad \sum_{c=b+1}^{a-1} \Delta_c \left(\overline{K}_c^{\frac{1}{2}}K_{b-1}^{\frac{1}{2}}E^a{}_c \otimes E^c{}_{b-1} K_a^{\frac{1}{2}}\overline{K}_c^{\frac{1}{2}}\right) \\
&= E^a{}_{b-1} \otimes K_a^{\frac{1}{2}}\overline{K}_{b-1}^{\frac{1}{2}} + \overline{K}_a^{\frac{1}{2}}K_{b-1}^{\frac{1}{2}} \otimes E^a{}_{b-1} - \\
&\qquad \sum_{c=(b-1)+1}^{a-1} \Delta_c \left(\overline{K}_c^{\frac{1}{2}}K_{b-1}^{\frac{1}{2}}E^a{}_c \otimes E^c{}_{b-1} K_a^{\frac{1}{2}}\overline{K}_c^{\frac{1}{2}}\right)
\end{aligned}
$$

Thus, the result is true for $a - b = p + 1$ if it is true for $a - b = p$, hence it is true for all lowering generators $E^a{}_b$, where $a > b$.

Now observe that the definition of Δ for simple raising generators is obtained from that for simple lowering generators by the mapping $q \mapsto \overline{q}$. Also, the definition of the nonsimple raising generators is obtained from that of the nonsimple lowering generators under the same mapping. Together, these definitions imply that the expression for Δ for the nonsimple raising generators may be obtained from that for Δ of the nonsimple lowering generators under that mapping $q \mapsto \overline{q}$.

$\square$

It is also possible to deduce the coproducts of nonsimple generators via an entirely different approach, using the operator $L(x)$ defined in [27], but we have not followed this up.

B Explicit results for the $U_q[gl(3|1)]$ case

Here, $[i] = 0$ for $i \in \{1; 5, 6, 7\}$ and $[i] = 1$ for $i \in \{2, 3, 4; 8\}$. Apart from all the notational conventions mentioned in the main text (recall $\Delta \triangleq q - \overline{q}$ of §5.2.2 and the $A_{i,j}^z$ and $C_{i,j}$ of (72) and (73) in §6.3), we add a couple more to condense the results.

- We use the following notation as a shorthand for the q graded symmetric combination of tensor product vectors:

$$q_{\pm}^x \, |i{\otimes}j\rangle \quad \triangleq \quad q^x \, |i{\otimes}j\rangle \pm \overline{q}^x \, |j{\otimes}i\rangle .$$

  This notation leads to such eyesores as "$q_{\pm}^0 \, |i{\otimes}j\rangle$". In these expressions, we shall choose $i \leqslant j$: observe that if $j > i$, we may replace $q_{\pm}^x \, |i{\otimes}j\rangle$ with $\pm q_{\pm}^{-x} \, |j{\otimes}i\rangle \equiv \pm \overline{q}_{\pm}^x \, |j{\otimes}i\rangle$.

- To convert the *graded* R matrices into the equivalent ungraded objects, simply multiply all terms in **boldface** by -1.

  Having done this conversion the following notation is a convenient shorthand for the q graded symmetric combination of rank 4 tensors in the resultant ungraded R matrices:

$$q_{\pm}^x e_{jl}^{ik} \quad \triangleq \quad q^x e_{jl}^{ik} \pm \overline{q}^x e_{lj}^{ki} .$$

B.1   Matrix elements of the $U_q[gl(3|1)]$ generators

For completeness, here we present matrix elements $\pi_{(0,0,0\,|\,\alpha)}(X)$ for *all* the $U_q[gl(3|1)]$ generators X (including even the $E^a{}_a$, for comparison with the K_a).

$$\pi(E^1{}_1) = -e^4_4 - e^6_6 - e^7_7 - e^8_8$$
$$\pi(E^2{}_2) = -e^3_3 - e^5_5 - e^7_7 - e^8_8$$
$$\pi(E^3{}_3) = -e^2_2 - e^5_5 - e^6_6 - e^8_8$$
$$\pi(K_1) = e^1_1 + e^2_2 + e^3_3 + \overline{q}e^4_4 + e^5_5 + \overline{q}e^6_6 + \overline{q}e^7_7 + \overline{q}e^8_8$$
$$\pi(K_2) = e^1_1 + e^2_2 + \overline{q}e^3_3 + e^4_4 + \overline{q}e^5_5 + e^6_6 + \overline{q}e^7_7 + \overline{q}e^8_8$$
$$\pi(K_3) = e^1_1 + \overline{q}e^2_2 + e^3_3 + e^4_4 + \overline{q}e^5_5 + \overline{q}e^6_6 + e^7_7 + \overline{q}e^8_8$$
$$\pi(E^4{}_4) = \alpha e^1_1 + (\alpha + 1)(e^2_2 + e^3_3 + e^4_4) + (\alpha + 2)(e^5_5 + e^6_6 + e^7_7) + (\alpha + 3)e^8_8$$
$$\pi(K_4) = \overline{q}^\alpha e^1_1 + \overline{q}^{\alpha+1}(e^2_2 + e^3_3 + e^4_4) + \overline{q}^{\alpha+2}(e^5_5 + e^6_6 + e^7_7) + \overline{q}^{\alpha+3}e^8_8$$
$$\pi(E^1{}_2) = -e^3_4 - e^5_6$$
$$\pi(E^2{}_1) = -e^4_3 - e^6_5$$
$$\pi(E^2{}_3) = -e^2_3 - e^6_7$$
$$\pi(E^3{}_2) = -e^3_2 - e^7_6$$
$$\pi(E^3{}_4) = A_0^{\frac{1}{2}}e^1_2 + A_1^{\frac{1}{2}}e^3_5 + A_1^{\frac{1}{2}}e^4_6 + A_2^{\frac{1}{2}}e^7_8$$
$$\pi(E^4{}_3) = A_0^{\frac{1}{2}}e^2_1 + A_1^{\frac{1}{2}}e^5_3 + A_1^{\frac{1}{2}}e^6_4 + A_2^{\frac{1}{2}}e^8_7$$
$$\pi(E^1{}_3) = e^5_7 - \overline{q}e^2_4$$
$$\pi(E^3{}_1) = e^7_5 - qe^4_2$$
$$\pi(E^2{}_4) = \overline{q}A_0^{\frac{1}{2}}e^1_3 - A_1^{\frac{1}{2}}e^2_5 + \overline{q}A_1^{\frac{1}{2}}e^4_7 - A_2^{\frac{1}{2}}e^6_8$$
$$\pi(E^4{}_2) = qA_0^{\frac{1}{2}}e^3_1 - A_1^{\frac{1}{2}}e^5_2 + qA_1^{\frac{1}{2}}e^7_4 - A_2^{\frac{1}{2}}e^8_6$$
$$\pi(E^1{}_4) = -\overline{q}^2 A_0^{\frac{1}{2}}e^1_4 - \overline{q}A_1^{\frac{1}{2}}e^2_6 - \overline{q}A_1^{\frac{1}{2}}e^3_7 + A_2^{\frac{1}{2}}e^5_8$$
$$\pi(E^4{}_1) = -q^2 A_0^{\frac{1}{2}}e^4_1 - qA_1^{\frac{1}{2}}e^6_2 - qA_1^{\frac{1}{2}}e^7_3 + A_2^{\frac{1}{2}}e^8_5$$

B.2   The basis $\mathfrak{B}_0$ for $V_0 \equiv V_{(0,0,0\,|\,2\alpha)}$

The 8 vectors in this basis are:

$$|1\otimes 1\rangle$$
$$\overline{C}_0^{\frac{1}{2}} q_+^{\frac{\alpha}{2}} \{|1\otimes 2\rangle, |1\otimes 3\rangle, |1\otimes 4\rangle\}$$
$$\overline{C}_0^{\frac{1}{2}} \overline{A}_{2,1}^{\frac{1}{2}} \begin{cases} A_1^{\frac{1}{2}} q_+^\alpha |1\otimes 5\rangle + A_0^{\frac{1}{2}} \overline{q}_-^{\frac{1}{2}} |2\otimes 3\rangle, \\ A_1^{\frac{1}{2}} q_+^\alpha |1\otimes 6\rangle + A_0^{\frac{1}{2}} \overline{q}_-^{\frac{1}{2}} |2\otimes 4\rangle, \\ A_1^{\frac{1}{2}} q_+^\alpha |1\otimes 7\rangle + A_0^{\frac{1}{2}} \overline{q}_-^{\frac{1}{2}} |3\otimes 4\rangle \end{cases}$$
$$\overline{C}_0^{\frac{1}{2}} \overline{C}_1^{\frac{1}{2}} \overline{A}_{2,1}^{\frac{1}{2}} \left[A_0^{\frac{1}{2}} \left(q_+^{\frac{\alpha}{2}} |3\otimes 6\rangle - q_+^{\frac{\alpha}{2}-1} |2\otimes 7\rangle - q_+^{\frac{\alpha}{2}+1} |4\otimes 5\rangle \right) - A_2^{\frac{1}{2}} q_+^{\frac{3\alpha}{2}} |1\otimes 8\rangle \right]$$

B.3   The basis $\mathfrak{B}_1$ for $V_1 \equiv V_{(0,0,-1\,|\,2\alpha+1)}$

The 24 vectors in this basis are:

$$\left\{\left|2\otimes2\right\rangle,\left|3\otimes3\right\rangle,\left|4\otimes4\right\rangle\right\}$$

$$\overline{C}_0^{\frac{1}{2}}\overline{q}_-^{\frac{\alpha}{2}}\left\{\left|1\otimes2\right\rangle,\left|1\otimes3\right\rangle,\left|1\otimes4\right\rangle\right\}$$

$$\overline{C}_1^{\frac{1}{2}}q_-^{\frac{\alpha}{2}+\frac{1}{2}}\left\{\left|2\otimes5\right\rangle,\left|3\otimes5\right\rangle,\left|2\otimes6\right\rangle,\left|3\otimes7\right\rangle,\left|4\otimes7\right\rangle,\left|4\otimes6\right\rangle\right\}$$

$$\overline{C}_0^{\frac{1}{2}}\overline{A}_{2,1}^{\frac{1}{2}}\left\{\begin{array}{l}A_1^{\frac{1}{2}}q_-^0\left|1\otimes5\right\rangle+A_0^{\frac{1}{2}}\overline{q}_+^{\alpha+\frac{1}{2}}\left|2\otimes3\right\rangle,\\A_1^{\frac{1}{2}}q_-^0\left|1\otimes6\right\rangle+A_0^{\frac{1}{2}}\overline{q}_+^{\alpha+\frac{1}{2}}\left|2\otimes4\right\rangle,\\A_1^{\frac{1}{2}}q_-^0\left|1\otimes7\right\rangle-A_0^{\frac{1}{2}}\overline{q}_+^{\alpha+\frac{1}{2}}\left|3\otimes4\right\rangle\end{array}\right\}$$

$$\overline{C}_1^{\frac{1}{2}}\overline{A}_{2,1}^{\frac{1}{2}}\left\{\begin{array}{l}A_0^{\frac{1}{2}}q_-^0\left|1\otimes6\right\rangle-A_1^{\frac{1}{2}}q_+^{\alpha+\frac{1}{2}}\left|2\otimes4\right\rangle,\\A_0^{\frac{1}{2}}q_-^0\left|1\otimes5\right\rangle-A_1^{\frac{1}{2}}q_+^{\alpha+\frac{1}{2}}\left|2\otimes3\right\rangle,\\A_0^{\frac{1}{2}}q_-^0\left|1\otimes7\right\rangle-A_1^{\frac{1}{2}}q_+^{\alpha+\frac{1}{2}}\left|3\otimes4\right\rangle\end{array}\right\}$$

$$\overline{C}_1^{\frac{1}{2}}\overline{A}_{2,3}^{\frac{1}{2}}\left\{\begin{array}{l}A_2^{\frac{1}{2}}q_+^{\alpha+1}\left|2\otimes8\right\rangle-A_1^{\frac{1}{2}}\overline{q}_-^{\frac{1}{2}}\left|5\otimes6\right\rangle,\\A_2^{\frac{1}{2}}q_+^{\alpha+1}\left|3\otimes8\right\rangle-A_1^{\frac{1}{2}}\overline{q}_-^{\frac{1}{2}}\left|5\otimes7\right\rangle,\\A_2^{\frac{1}{2}}q_+^{\alpha+1}\left|4\otimes8\right\rangle-A_1^{\frac{1}{2}}\overline{q}_-^{\frac{1}{2}}\left|6\otimes7\right\rangle\end{array}\right\}$$

$$\overline{C}_0^{\frac{1}{2}}\overline{C}_1^{\frac{1}{2}}\overline{A}_{2,1}^{\frac{1}{2}}\left[A_2^{\frac{1}{2}}q_-^{\frac{\alpha}{2}}\left|1\otimes8\right\rangle+A_0^{\frac{1}{2}}\overline{q}_-^{\frac{\alpha}{2}+1}\left|2\otimes7\right\rangle-A_0^{\frac{1}{2}}\overline{q}_-^{\frac{\alpha}{2}}\left|3\otimes6\right\rangle-A_0^{\frac{1}{2}}q_-^{\frac{3\alpha}{2}+1}\left|4\otimes5\right\rangle\right]$$

$$\overline{C}_1\overline{A}_1^{\frac{1}{2}}\overline{A}_{2,1}^{\frac{1}{2}}\left[A_0^{\frac{1}{2}}A_2^{\frac{1}{2}}q_-^{\frac{\alpha}{2}}\left|1\otimes8\right\rangle+A_0\overline{q}_-^{\frac{\alpha}{2}+1}\left|2\otimes7\right\rangle+A_1q_-^{\frac{3\alpha}{2}+1}\left|3\otimes6\right\rangle+A_1\overline{q}_-^{\frac{\alpha}{2}}\left|4\otimes5\right\rangle\right]$$

$$\overline{C}_1\overline{A}_1^{\frac{1}{2}}\overline{A}_{2,3}^{\frac{1}{2}}\left[A_0^{\frac{1}{2}}A_2^{\frac{1}{2}}q_-^{\frac{\alpha}{2}}\left|1\otimes8\right\rangle-A_2q_-^{\frac{3\alpha}{2}+1}\left|2\otimes7\right\rangle-A_1\overline{q}_-^{\frac{\alpha}{2}+1}\left|3\otimes6\right\rangle+A_1\overline{q}_-^{\frac{\alpha}{2}}\left|4\otimes5\right\rangle\right]$$

B.4   The basis $\mathfrak{B}_2$ for $V_2 \equiv V_{(0,-1,-1\,|\,2\alpha+2)}$

The 24 vectors in this basis are:

$$\overline{C}_1\overline{A}_1^{\frac{1}{2}}\overline{A}_{2,1}^{\frac{1}{2}}\left[A_0^{\frac{1}{2}}A_2^{\frac{1}{2}}\overline{q}_+^{\frac{\alpha}{2}+1}\left|1\otimes8\right\rangle+A_0\overline{q}_+^{\frac{3\alpha}{2}+2}\left|2\otimes7\right\rangle+A_1q_+^{\frac{\alpha}{2}}\left|3\otimes6\right\rangle-A_1q_+^{\frac{\alpha}{2}+1}\left|4\otimes5\right\rangle\right]$$

$$\overline{C}_1\overline{A}_1^{\frac{1}{2}}\overline{A}_{2,3}^{\frac{1}{2}}\left[A_0^{\frac{1}{2}}A_2^{\frac{1}{2}}\overline{q}_+^{\frac{\alpha}{2}+1}\left|1\otimes8\right\rangle-A_2q_+^{\frac{\alpha}{2}}\left|2\otimes7\right\rangle-A_1\overline{q}_+^{\frac{3\alpha}{2}+2}\left|3\otimes6\right\rangle-A_1q_+^{\frac{\alpha}{2}+1}\left|4\otimes5\right\rangle\right]$$

$$\overline{C}_1^{\frac{1}{2}}\overline{C}_2^{\frac{1}{2}}\overline{A}_{2,3}^{\frac{1}{2}}\left[A_0^{\frac{1}{2}}\overline{q}_+^{\frac{\alpha}{2}+1}\left|1\otimes8\right\rangle-A_2^{\frac{1}{2}}q_+^{\frac{\alpha}{2}}\left|2\otimes7\right\rangle+A_2^{\frac{1}{2}}q_+^{\frac{\alpha}{2}+1}\left|3\otimes6\right\rangle+A_2^{\frac{1}{2}}\overline{q}_+^{\frac{3\alpha}{2}+2}\left|4\otimes5\right\rangle\right]$$

$$\overline{C}_1^{\frac{1}{2}}\overline{A}_{2,1}^{\frac{1}{2}}\left\{\begin{array}{l}A_0^{\frac{1}{2}}\overline{q}_+^{\alpha+1}\left|1\otimes5\right\rangle-A_1^{\frac{1}{2}}\overline{q}_-^{\frac{1}{2}}\left|2\otimes3\right\rangle,\\A_0^{\frac{1}{2}}\overline{q}_+^{\alpha+1}\left|1\otimes6\right\rangle-A_1^{\frac{1}{2}}\overline{q}_-^{\frac{1}{2}}\left|2\otimes4\right\rangle,\\A_0^{\frac{1}{2}}\overline{q}_+^{\alpha+1}\left|1\otimes7\right\rangle-A_1^{\frac{1}{2}}\overline{q}_-^{\frac{1}{2}}\left|3\otimes4\right\rangle\end{array}\right\}$$

$$\overline{C}_1^{\frac{1}{2}}\overline{A}_{2,3}^{\frac{1}{2}}\left\{\begin{array}{l}A_2^{\frac{1}{2}}q_-^0\left|2\otimes8\right\rangle-A_1^{\frac{1}{2}}\overline{q}_+^{\alpha+\frac{3}{2}}\left|5\otimes6\right\rangle,\\A_2^{\frac{1}{2}}q_-^0\left|3\otimes8\right\rangle-A_1^{\frac{1}{2}}\overline{q}_+^{\alpha+\frac{3}{2}}\left|5\otimes7\right\rangle,\\A_2^{\frac{1}{2}}q_-^0\left|4\otimes8\right\rangle-A_1^{\frac{1}{2}}\overline{q}_+^{\alpha+\frac{3}{2}}\left|6\otimes7\right\rangle\end{array}\right\}$$

$$\overline{C}_2^{\frac{1}{2}}\overline{A}_{2,3}^{\frac{1}{2}}\left\{\begin{array}{l}A_1^{\frac{1}{2}}q_-^0\left|3\otimes8\right\rangle+A_2^{\frac{1}{2}}q_+^{\alpha+\frac{3}{2}}\left|5\otimes7\right\rangle,\\A_1^{\frac{1}{2}}q_-^0\left|4\otimes8\right\rangle+A_2^{\frac{1}{2}}q_+^{\alpha+\frac{3}{2}}\left|6\otimes7\right\rangle,\\A_1^{\frac{1}{2}}q_-^0\left|2\otimes8\right\rangle+A_2^{\frac{1}{2}}q_+^{\alpha+\frac{3}{2}}\left|5\otimes6\right\rangle\end{array}\right\}$$

$$\overline{C}_1^{\frac{1}{2}}\overline{q}_+^{\frac{\alpha}{2}+\frac{1}{2}}\left\{\left|2\otimes6\right\rangle,\left|2\otimes5\right\rangle,\left|3\otimes5\right\rangle,\left|3\otimes7\right\rangle,\left|4\otimes6\right\rangle,\left|4\otimes7\right\rangle\right\}$$

$$\overline{C}_2^{\frac{1}{2}}q_+^{\frac{\alpha}{2}+1}\left\{\left|5\otimes8\right\rangle,\left|7\otimes8\right\rangle,\left|6\otimes8\right\rangle\right\}$$

$$\left\{\left|7\otimes7\right\rangle,\left|6\otimes6\right\rangle,\left|5\otimes5\right\rangle\right\}$$

B.5   The basis $\mathfrak{B}_3$ for $V_3 \equiv V_{(-1,-1,-1\,|\,2\alpha+3)}$

The 8 vectors in this basis are:

$$\overline{C}_1^{\frac{1}{2}}\overline{C}_2^{\frac{1}{2}}\overline{A}_{2,3}^{\frac{1}{2}}\left[A_0^{\frac{1}{2}}\overline{q}_-^{\frac{3\alpha}{2}+3}\,|1\otimes 8\rangle - A_2^{\frac{1}{2}}\left(\overline{q}_-^{\frac{\alpha}{2}+2}\,|2\otimes 7\rangle - \overline{q}_-^{\frac{\alpha}{2}+1}\,|3\otimes 6\rangle + \overline{q}_-^{\frac{\alpha}{2}}\,|4\otimes 5\rangle\right)\right]$$

$$\overline{C}_2^{\frac{1}{2}}\overline{A}_{2,3}^{\frac{1}{2}}\left\{\begin{array}{l}A_1^{\frac{1}{2}}\overline{q}_+^{\alpha+2}\,|2\otimes 8\rangle + A_2^{\frac{1}{2}}\overline{q}_-^{\frac{1}{2}}\,|5\otimes 6\rangle\,,\\[4pt]A_1^{\frac{1}{2}}\overline{q}_+^{\alpha+2}\,|3\otimes 8\rangle + A_2^{\frac{1}{2}}\overline{q}_-^{\frac{1}{2}}\,|5\otimes 7\rangle\,,\\[4pt]A_1^{\frac{1}{2}}\overline{q}_+^{\alpha+2}\,|4\otimes 8\rangle + A_2^{\frac{1}{2}}\overline{q}_-^{\frac{1}{2}}\,|6\otimes 7\rangle\end{array}\right\}$$

$$\overline{C}_2^{\frac{1}{2}}\overline{q}_-^{\frac{\alpha}{2}+1}\left\{|5\otimes 8\rangle\,,|6\otimes 8\rangle\,,|7\otimes 8\rangle\right\}$$

$$|8\otimes 8\rangle$$

B.6   The trigonometric R matrix $\check{R}^{3,1}(u)$

For the listing of the components of $\check{R}^{3,1}(u)$, we invoke a little more notation:

$$\begin{aligned}S_i^{\pm} &\triangleq [\alpha + i \pm u]_q,\\S_i^z &\triangleq [u-i]_q^{\;z}, \qquad \text{where } z \in \{1,2\},\end{aligned}$$

and $i \in \{0,1,2\}$. With this, $\check{R}^{3,1}(u)$ has 216 nonzero components:

$$1\left\{e_{11}^{11}\right\},\qquad \frac{S_0^+}{S_0^-}\left\{e_{22}^{22},e_{33}^{33},e_{44}^{44}\right\},\qquad \frac{S_0^+ S_1^+}{S_0^- S_1^-}\left\{e_{55}^{55},e_{66}^{66},e_{77}^{77}\right\},\qquad \frac{S_0^+ S_1^+ S_2^+}{S_0^- S_1^- S_2^-}\left\{e_{88}^{88}\right\}$$

$$\frac{A_0}{S_0^-}\left\{\begin{array}{l}\overline{q}^u\left\{e_{12}^{12},e_{13}^{13},e_{14}^{14}\right\}\\ q^u\left\{e_{21}^{21},e_{31}^{31},e_{41}^{41}\right\}\end{array}\right\},\qquad \frac{A_2 S_0^+ S_1^+}{S_0^- S_1^- S_2^-}\left\{\begin{array}{l}q^u\left\{e_{87}^{87},e_{86}^{86},e_{85}^{85}\right\}\\ \overline{q}^u\left\{e_{78}^{78},e_{68}^{68},e_{58}^{58}\right\}\end{array}\right\}$$

$$\frac{1}{\Delta^2 S_0^- S_1^-}\left\{\begin{array}{l}f_1(q)\left\{e_{23}^{23},e_{24}^{24},e_{34}^{34}\right\}\\ f_1(\overline{q})\left\{e_{32}^{32},e_{42}^{42},e_{43}^{43}\right\}\end{array}\right\},\qquad \frac{1}{\Delta^2 S_0^- S_1^- S_2^-}\left\{\begin{array}{l}f_2(q)\left\{e_{76}^{76},e_{75}^{75},e_{65}^{65}\right\}\\ f_2(\overline{q})\left\{e_{67}^{67},e_{57}^{57},e_{56}^{56}\right\}\end{array}\right\}$$

$$\frac{A_0 A_1}{S_0^- S_1^-}\left\{\begin{array}{l}q^{2u}\left\{e_{51}^{51},e_{61}^{61},e_{71}^{71}\right\}\\ \overline{q}^{2u}\left\{e_{15}^{15},e_{16}^{16},e_{17}^{17}\right\}\end{array}\right\},\qquad \frac{A_1 A_2 S_0^+}{S_0^- S_1^- S_2^-}\left\{\begin{array}{l}q^{2u}\left\{e_{84}^{84},e_{83}^{83},e_{82}^{82}\right\}\\ \overline{q}^{2u}\left\{e_{48}^{48},e_{38}^{38},e_{28}^{28}\right\}\end{array}\right\}$$

$$\frac{A_1}{\Delta^2 S_0^- S_1^- S_2^-}\left\{\begin{array}{l}f_3(q)\left\{e_{27}^{27}\right\},f_4(q)\left\{e_{36}^{36}\right\},f_5(q)\left\{e_{45}^{45}\right\}\\ f_3(\overline{q})\left\{e_{72}^{72}\right\},f_4(\overline{q})\left\{e_{63}^{63}\right\},f_5(\overline{q})\left\{e_{54}^{54}\right\}\end{array}\right\}$$

$$\frac{A_1 S_0^+}{S_0^- S_1^-}\left\{\begin{array}{l}q^u\left\{e_{52}^{52},e_{53}^{53},e_{62}^{62},e_{64}^{64},e_{73}^{73},e_{74}^{74}\right\}\\ \overline{q}^u\left\{e_{25}^{25},e_{35}^{35},e_{26}^{26},e_{46}^{46},e_{37}^{37},e_{47}^{47}\right\}\end{array}\right\},\qquad \frac{A_0 A_1 A_2}{S_0^- S_1^- S_2^-}\left\{\begin{array}{l}q^{3u}\left\{e_{81}^{81}\right\}\\ \overline{q}^{3u}\left\{e_{18}^{18}\right\}\end{array}\right\}$$

$$\frac{U_0}{S_0^-}\left\{\begin{array}{l}+\mathbf{1}\left\{e_{21}^{12},e_{31}^{13},e_{41}^{14}\right\}\\ -\mathbf{1}\left\{e_{12}^{21},e_{13}^{31},e_{14}^{41}\right\}\end{array}\right\},\quad \frac{U_0 S_0^+ S_1^+}{S_0^- S_1^- S_2^-}\left\{\begin{array}{l}+\mathbf{1}\left\{e_{87}^{78},e_{86}^{68},e_{85}^{58}\right\}\\ -\mathbf{1}\left\{e_{78}^{87},e_{68}^{86},e_{58}^{85}\right\}\end{array}\right\},\quad \frac{U_0 U_1 U_2}{S_0^- S_1^- S_2^-}\left\{\begin{array}{l}+\mathbf{1}\left\{e_{81}^{18}\right\}\\ -\mathbf{1}\left\{e_{18}^{81}\right\}\end{array}\right\}$$

$$\frac{U_0 U_1}{S_0^- S_1^-}\left\{\begin{array}{l}e_{51}^{15},e_{61}^{16},e_{71}^{17}\\ e_{15}^{51},e_{16}^{61},e_{17}^{71}\end{array}\right\},\qquad -\frac{U_0 U_1 S_0^+}{S_0^- S_1^- S_2^-}\left\{\begin{array}{l}e_{48}^{84},e_{38}^{83},e_{28}^{82}\\ e_{84}^{48},e_{83}^{38},e_{82}^{28}\end{array}\right\}$$

$$-\frac{U_0^2}{S_0^- S_1^-}\left\{\begin{array}{l}e_{32}^{23},e_{42}^{24},e_{43}^{34}\\ e_{23}^{32},e_{24}^{42},e_{34}^{43}\end{array}\right\},\qquad -\frac{U_0^2 S_0^+}{S_0^- S_1^- S_2^-}\left\{\begin{array}{l}e_{67}^{76},e_{57}^{75},e_{56}^{65}\\ e_{76}^{67},e_{75}^{57},e_{65}^{56}\end{array}\right\}$$

$$\frac{U_0^2 U_1}{S_0^- S_1^- S_2^-}\left\{\begin{array}{l}+\mathbf{1}\left\{e_{45}^{54},e_{36}^{63},e_{27}^{72}\right\}\\ -\mathbf{1}\left\{e_{54}^{45},e_{63}^{36},e_{72}^{27}\right\}\end{array}\right\},\qquad \frac{U_0 S_0^+}{S_0^- S_1^-}\left\{\begin{array}{l}-\mathbf{1}\left\{e_{25}^{52},e_{35}^{53},e_{26}^{62},e_{46}^{64},e_{37}^{73},e_{47}^{74}\right\}\\ +\mathbf{1}\left\{e_{52}^{25},e_{53}^{35},e_{62}^{26},e_{64}^{46},e_{73}^{37},e_{74}^{47}\right\}\end{array}\right\}$$

$$\frac{A_0^{\frac{1}{2}}A_1^{\frac{1}{2}}U_0}{S_0^-S_1^-}\left\{\begin{matrix}q^{u+\frac{1}{2}}\left\{\begin{matrix}-\mathbf{1}\left\{e_{32}^{51},e_{42}^{61},e_{43}^{71}\right\}\\+1\left\{e_{51}^{32},e_{61}^{42},e_{71}^{43}\right\}\end{matrix}\right\}\\q^{u-\frac{1}{2}}\left\{\begin{matrix}+\mathbf{1}\left\{e_{23}^{51},e_{24}^{61},e_{34}^{71}\right\}\\-1\left\{e_{51}^{23},e_{61}^{24},e_{71}^{34}\right\}\end{matrix}\right\}\\\overline{q}^{u+\frac{1}{2}}\left\{\begin{matrix}+\mathbf{1}\left\{e_{23}^{15},e_{24}^{16},e_{34}^{17}\right\}\\-1\left\{e_{15}^{23},e_{16}^{24},e_{17}^{34}\right\}\end{matrix}\right\}\\\overline{q}^{u-\frac{1}{2}}\left\{\begin{matrix}-\mathbf{1}\left\{e_{32}^{15},e_{42}^{16},e_{43}^{17}\right\}\\+1\left\{e_{15}^{32},e_{16}^{42},e_{17}^{43}\right\}\end{matrix}\right\}\end{matrix}\right\},\ \frac{A_1^{\frac{1}{2}}A_2^{\frac{1}{2}}U_0S_0^+}{S_0^-S_1^-S_2^-}\left\{\begin{matrix}q^{u+\frac{1}{2}}\left\{\begin{matrix}+\mathbf{1}\left\{e_{82}^{65},e_{83}^{75},e_{84}^{76}\right\}\\-1\left\{e_{65}^{82},e_{75}^{83},e_{76}^{84}\right\}\end{matrix}\right\}\\q^{u-\frac{1}{2}}\left\{\begin{matrix}-\mathbf{1}\left\{e_{82}^{56},e_{83}^{57},e_{84}^{67}\right\}\\+1\left\{e_{56}^{82},e_{57}^{83},e_{67}^{84}\right\}\end{matrix}\right\}\\\overline{q}^{u+\frac{1}{2}}\left\{\begin{matrix}-\mathbf{1}\left\{e_{28}^{56},e_{38}^{57},e_{48}^{67}\right\}\\+1\left\{e_{56}^{28},e_{57}^{38},e_{67}^{48}\right\}\end{matrix}\right\}\\\overline{q}^{u-\frac{1}{2}}\left\{\begin{matrix}+\mathbf{1}\left\{e_{28}^{65},e_{38}^{75},e_{48}^{76}\right\}\\-1\left\{e_{65}^{28},e_{75}^{38},e_{76}^{48}\right\}\end{matrix}\right\}\end{matrix}\right\}$$

$$\frac{U_0}{\Delta^2 S_0^-S_1^-S_2^-}\left\{f_6(q)\left\{\begin{matrix}+\mathbf{1}\left\{e_{45}^{63},-qe_{45}^{72},e_{36}^{72}\right\}\\-1\left\{e_{63}^{45},-qe_{72}^{45},e_{72}^{36}\right\}\end{matrix}\right\},\ f_6(\overline{q})\left\{\begin{matrix}+\mathbf{1}\left\{e_{36}^{54},-\overline{q}e_{27}^{54},e_{27}^{63}\right\}\\-1\left\{e_{54}^{36},-\overline{q}e_{54}^{27},e_{63}^{27}\right\}\end{matrix}\right\}\right\}$$

$$\frac{A_1U_0^2}{S_0^-S_1^-S_2^-}\left\{\begin{matrix}\overline{q}^u\left\{-q\left\{\begin{matrix}e_{45}^{36}\\e_{36}^{45}\end{matrix}\right\},\left\{\begin{matrix}e_{45}^{27}\\e_{27}^{45}\end{matrix}\right\},-\overline{q}\left\{\begin{matrix}e_{27}^{36}\\e_{36}^{27}\end{matrix}\right\}\right\}\\q^u\left\{-\overline{q}\left\{\begin{matrix}e_{54}^{63}\\e_{63}^{54}\end{matrix}\right\},\left\{\begin{matrix}e_{54}^{72}\\e_{72}^{54}\end{matrix}\right\},-q\left\{\begin{matrix}e_{72}^{63}\\e_{63}^{72}\end{matrix}\right\}\right\}\end{matrix}\right\}$$

$$\frac{A_0^{\frac{1}{2}}A_2^{\frac{1}{2}}U_0U_1}{S_0^-S_1^-S_2^-}\left\{\begin{matrix}\overline{q}^u\left\{q\left\{\begin{matrix}e_{72}^{18}\\e_{18}^{72}\end{matrix}\right\},-\left\{\begin{matrix}e_{63}^{18}\\e_{18}^{63}\end{matrix}\right\},\overline{q}\left\{\begin{matrix}e_{54}^{18}\\e_{18}^{54}\end{matrix}\right\}\right\}\\q^u\left\{\overline{q}\left\{\begin{matrix}e_{27}^{81}\\e_{81}^{27}\end{matrix}\right\},-\left\{\begin{matrix}e_{36}^{81}\\e_{81}^{36}\end{matrix}\right\},q\left\{\begin{matrix}e_{45}^{81}\\e_{81}^{45}\end{matrix}\right\}\right\}\end{matrix}\right\}$$

$$\frac{A_0^{\frac{1}{2}}A_1A_2^{\frac{1}{2}}U_0}{S_0^-S_1^-S_2^-}\left\{\begin{matrix}\overline{q}^{2u}\left\{\overline{q}\left\{\begin{matrix}+\mathbf{e_{27}^{18}}\\-e_{18}^{27}\end{matrix}\right\},-\left\{\begin{matrix}+\mathbf{e_{36}^{18}}\\-e_{18}^{36}\end{matrix}\right\},q\left\{\begin{matrix}+\mathbf{e_{45}^{18}}\\-e_{18}^{45}\end{matrix}\right\}\right\}\\q^{2u}\left\{q\left\{\begin{matrix}+\mathbf{e_{81}^{72}}\\-e_{72}^{81}\end{matrix}\right\},-\left\{\begin{matrix}+\mathbf{e_{81}^{63}}\\-e_{63}^{81}\end{matrix}\right\},\overline{q}\left\{\begin{matrix}+\mathbf{e_{81}^{54}}\\-e_{54}^{81}\end{matrix}\right\}\right\}\end{matrix}\right\},$$

where:

$$
\begin{aligned}
f_1(q) &= -2\overline{q}+(q^{1+2\alpha}+\overline{q}^{1+2\alpha})-\overline{q}^{2u}(q-\overline{q})\\
f_2(q) &= -2q+(\overline{q}^{3+2\alpha}+q^{3+2\alpha})+q^{2u}(q-\overline{q})\\
f_3(q) &= -\overline{q}^u(2\overline{q}^2-(q^{2+2\alpha}+\overline{q}^{2+2\alpha})+\overline{q}^{2u}(q^2-\overline{q}^2))\\
f_4(q) &= \overline{q}^u(-2+(q^{2+2\alpha}+\overline{q}^{2+2\alpha})-(q^{2-2u}+\overline{q}^{2-2u})+(q^{2u}+\overline{q}^{2u}))\\
f_5(q) &= \overline{q}^u(-2q^2+(q^{2+2\alpha}+\overline{q}^{2+2\alpha})+q^{2u}(q^2-\overline{q}^2))\\
f_6(q) &= q(q+\overline{q})-(q^{2+2\alpha}+\overline{q}^{2+2\alpha})-q^{2u-1}(q-\overline{q}).
\end{aligned}
$$

B.7 The quantum R matrix $\check{R}^{3,1}$

$\check{R}^{3,1}$ has 139 nonzero components:

$$1\left\{e_{11}^{11}\right\}, \qquad -q^{2\alpha}\left\{e_{22}^{22}, e_{33}^{33}, e_{44}^{44},\right\}, \qquad q^{4\alpha+2}\left\{e_{55}^{55}, e_{66}^{66}, e_{77}^{77}\right\}, \qquad -q^{6\alpha+6}\left\{e_{88}^{88}\right\}$$

$$-\Delta q^{\alpha}A_0\left\{e_{21}^{21}, e_{31}^{31}, e_{41}^{41}\right\}, \qquad -\Delta q^{5\alpha+4}A_2\left\{e_{87}^{87}, e_{86}^{86}, e_{85}^{85}\right\}$$

$$\Delta q^{2\alpha+1}\left\{e_{32}^{32}, e_{42}^{42}, e_{43}^{43}\right\}, \qquad -\Delta q^{4\alpha+3}\left\{e_{76}^{76}, e_{75}^{75}, e_{65}^{65}\right\}$$

$$\Delta^2 q^{2\alpha+1}A_0 A_1\left\{e_{51}^{51}, e_{61}^{61}, e_{71}^{71}\right\}, \qquad -\Delta^2 q^{4\alpha+3}A_1 A_2\left\{e_{84}^{84}, e_{83}^{83}, e_{82}^{82}\right\}$$

$$-\Delta^2 q^{3\alpha+2}A_1\left\{e_{63}^{63}\right\}, \qquad -\Delta q^{3\alpha+3}A_1(q^2-\overline{q}^2)\left\{e_{72}^{72}\right\}$$

$$\Delta q^{3\alpha+1}A_1\left\{e_{52}^{52}, e_{53}^{53}, e_{62}^{62}, e_{64}^{64}, e_{73}^{73}, e_{74}^{74}\right\}, \qquad -\Delta^3 q^{3\alpha+3}A_0 A_1 A_2\left\{e_{81}^{81}\right\}$$

$$q^{\alpha}\begin{Bmatrix} -\mathbf{1}\left\{e_{21}^{12}, e_{31}^{13}, e_{41}^{14}\right\} \\ +1\left\{e_{12}^{21}, e_{13}^{31}, e_{14}^{41}\right\} \end{Bmatrix}, \qquad q^{5\alpha+4}\begin{Bmatrix} -\mathbf{1}\left\{e_{87}^{78}, e_{86}^{68}, e_{85}^{58}\right\} \\ +1\left\{e_{78}^{87}, e_{68}^{86}, e_{58}^{85}\right\} \end{Bmatrix}$$

$$q^{2\alpha}\begin{Bmatrix} e_{51}^{15}, e_{61}^{16}, e_{71}^{17} \\ e_{15}^{51}, e_{16}^{61}, e_{17}^{71} \end{Bmatrix}, \qquad -q^{4\alpha+2}\begin{Bmatrix} e_{48}^{84}, e_{38}^{83}, e_{28}^{82} \\ e_{84}^{48}, e_{83}^{38}, e_{82}^{28} \end{Bmatrix}$$

$$-q^{2\alpha+1}\begin{Bmatrix} e_{32}^{23}, e_{42}^{24}, e_{43}^{34} \\ e_{23}^{32}, e_{24}^{42}, e_{34}^{43} \end{Bmatrix}, \qquad q^{4\alpha+3}\begin{Bmatrix} e_{67}^{76}, e_{57}^{75}, e_{56}^{65} \\ e_{76}^{67}, e_{75}^{57}, e_{65}^{56} \end{Bmatrix}$$

$$q^{3\alpha+1}\begin{Bmatrix} -\mathbf{1}\left\{e_{25}^{52}, e_{26}^{62}, e_{35}^{53}, e_{37}^{73}, e_{46}^{64}, e_{47}^{74}\right\} \\ +1\left\{e_{52}^{25}, e_{62}^{26}, e_{53}^{35}, e_{73}^{37}, e_{64}^{46}, e_{74}^{47}\right\} \end{Bmatrix}, q^{3\alpha+2}\begin{Bmatrix} -\mathbf{1}\left\{e_{27}^{72}, e_{36}^{63}, e_{45}^{54}\right\} \\ +1\left\{e_{72}^{27}, e_{63}^{36}, e_{54}^{45}\right\} \end{Bmatrix}, q^{3\alpha}\begin{Bmatrix} -\mathbf{1}\left\{e_{81}^{18}\right\} \\ +1\left\{e_{18}^{81}\right\} \end{Bmatrix}$$

$$\Delta q^{2\alpha+1}A_0 A_1\left\{\overline{q}^{\frac{1}{2}}\begin{Bmatrix} +\mathbf{1}\left\{e_{23}^{51}, e_{24}^{61}, e_{34}^{71}\right\} \\ -1\left\{e_{51}^{23}, e_{61}^{24}, e_{71}^{34}\right\} \end{Bmatrix}, q^{\frac{1}{2}}\begin{Bmatrix} -\mathbf{1}\left\{e_{32}^{51}, e_{42}^{61}, e_{43}^{71}\right\} \\ +1\left\{e_{51}^{32}, e_{61}^{42}, e_{71}^{43}\right\} \end{Bmatrix}\right\}$$

$$\Delta q^{4\alpha+3}A_1 A_2\left\{\overline{q}^{\frac{1}{2}}\begin{Bmatrix} +\mathbf{1}\left\{e_{82}^{56}, e_{83}^{57}, e_{84}^{67}\right\} \\ -1\left\{e_{56}^{82}, e_{57}^{83}, e_{67}^{84}\right\} \end{Bmatrix}, q^{\frac{1}{2}}\begin{Bmatrix} -\mathbf{1}\left\{e_{82}^{65}, e_{83}^{75}, e_{84}^{76}\right\} \\ +1\left\{e_{65}^{82}, e_{75}^{83}, e_{76}^{84}\right\} \end{Bmatrix}\right\}$$

$$\Delta q^{3\alpha+\frac{5}{2}}\left\{\overline{q}^{\frac{1}{2}}\begin{Bmatrix} +\mathbf{1}\left\{e_{45}^{63}, e_{36}^{72}\right\} \\ -1\left\{e_{63}^{45}, e_{72}^{36}\right\} \end{Bmatrix}, q^{\frac{1}{2}}\begin{Bmatrix} -\mathbf{1}\left\{e_{45}^{72}\right\} \\ +1\left\{e_{72}^{45}\right\} \end{Bmatrix}\right\}$$

$$\Delta q^{3\alpha+3}A_1\left\{\overline{q}\begin{Bmatrix} e_{54}^{63} \\ e_{63,}^{54} \end{Bmatrix}, -\begin{Bmatrix} e_{54}^{72} \\ e_{72}^{54} \end{Bmatrix}, q\begin{Bmatrix} e_{72}^{63} \\ e_{63}^{72} \end{Bmatrix}\right\}$$

$$\Delta q^{3\alpha+2}A_0 A_2\left\{-\overline{q}\begin{Bmatrix} e_{27}^{81} \\ e_{81}^{27} \end{Bmatrix}, \begin{Bmatrix} e_{36}^{81} \\ e_{81}^{36} \end{Bmatrix} - q\begin{Bmatrix} e_{45}^{81} \\ e_{81}^{45} \end{Bmatrix}\right\}$$

$$\Delta^2 q^{3\alpha+3}A_0 A_1 A_2\left\{\overline{q}\begin{Bmatrix} -\mathbf{e_{81}^{54}} \\ +e_{54}^{81} \end{Bmatrix}, -\begin{Bmatrix} -\mathbf{e_{81}^{63}} \\ +e_{63}^{81} \end{Bmatrix}, q\begin{Bmatrix} -\mathbf{e_{81}^{72}} \\ +e_{72}^{81} \end{Bmatrix}\right\}.$$

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